THE INITIAL VALUE PROBLEM FOR THE EQUATIONS OF MAGNETO-MICROPOLAR FLUID IN A TIME-DEPENDENT DOMAIN

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Abstract

In this work we study the equations of the mechanics of magneto-micropolar fluids in a time-dependent domain. By using the spectral Galerkin method together with the energy method and compactness arguments, we prove the existence of weak solutions.

Resumo

Neste trabalho estudamos as equações da mecânica de fluidos magneto-micropolar em um domínio dependendo do tempo. Usando o método de Galerkin espectral junto com o método de energia e argumentos de compacidade, provamos a existência de soluções fracas.

1. Introduction

The domain occupied by the fluid at time $t \in (0,T)$, $0 < T < \infty$, is denoted by $\Omega_t \subset \mathbb{R}^3$. We set $Q = \bigcup\limits_{0<t<T} \Omega_t \times \{t\} \subset \mathbb{R}^3 \times (0,T)$. whose lateral boundary is $\partial Q = \bigcup_{0<t<T} \partial\Omega_t \times \{t\}$. Let $u(x,t) \in \mathbb{R}^3, w(x,t) \in \mathbb{R}^3, b(x,t) \in \mathbb{R}^3$ and $p(x,t) \in \mathbb{R}$, denotes for $(x,t) \in Q$, respectively, the unknown velocity, the microrotational velocity, the magnetic field and the hydrostatic pressure of the fluid. Then, the governing equations are

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\begin{aligned}
&\frac{\partial u}{\partial t} + u.\nabla u - (\mu+\chi)\Delta u + \nabla(p + \frac{1}{2} r b.b) = \chi \text{rot } w + rb.\nabla b + f \\
&j\frac{\partial w}{\partial t} + ju.\nabla w - \gamma\Delta w + 2\chi w - (\alpha+\beta)\nabla \text{ div } w = \chi \text{rot } u + g \\
&\frac{\partial b}{\partial t} - \nu\Delta b + u.\nabla b - b.\nabla u = 0 \\
&\text{div } u = \text{ div } b = 0 \text{ in } Q.
\end{aligned} \tag{1.1}
$$

together with suitable boundary and initial conditions.

In this paper we will consider the problem of existence of weak solutions for that problem (1.1) in a time-dependent domain of $\mathbb{R}^3 \times (0,T), 0 < T < \infty$.

To (1.1) we append the following boundary and initial conditions:

$$
u(x,t) = w(x,t) = b(x,t) = 0, \quad \forall\, (x,t) \in \partial Q, \tag{1.2}
$$

$$
u(0) = u_0, \quad w(0) = w_0 \quad \text{and} \quad b(0) = b_0, \quad \forall\, x \in \Omega_0. \tag{1.3}
$$

where u_0, w_0 and b_0 are given functions. In (1.1), the differential operator ∇, Δ, div and rot are the usual gradient, Laplace, divergence and curl operators, respectively. The constants $\mu, \chi, r, \alpha, \beta, \gamma, j$ and ν are constants associated to properties of the material. From physical reasons, these constants satisfy $\min\{\mu, \chi, r, j, \nu, \alpha+\beta+\gamma\} > 0$; $f(x,t)$ and $g(x,t)$ are given external fields.

For the derivation and physical discussion of equations (1.1) - (1.3) see Condiff and Dalher [3], Eringen [5], [6], Ahmadi and Shanbinpoor [1], for instance. Equations (1.1) (i) has the familiar form of the Navier-Stokes, equations but is coupled with equation (1.1) (ii), which essentially describes the motion inside the macrovolumes as they undergo microrotational effects represented by the microrotational velocity vector w. For fluids with no microstructure this parameter vanishes. For Newtonian fluids, equation (1.1) (i) e (1.1) (ii) decouple since $\chi = 0$.

It is appropriate to cite some earlier works on the initial - value problem (1.1) - (1.3) which are related to ours and also locate our contribution therein. In cylindrical domain and when the magnetic field is absent ($b \equiv 0$), the reduced problem was studied by Lukaszewicz [11], [12]. Lukaszewicz [11] stablished the global existence of weak solutions for (1.1) - (1.3) under certain assumptions by using linearization and an almost fixed point theorem. In the same case, by using the same technique, Lukaszewicz [12] also proved the local and global existence, as well as the uniqueness of strong solutions. Again when $b \equiv 0$, Galdi and Rionero [8] stablished results similar to the ones of Lukaszewicz [12].

The full systems (1.1)-(1.3) in the cylindrical case, was studied by Galdi and Rionero [8] and they stated without proofs of existence and uniqueness of strong solutions. Rojas-Medar [19], Ortega-Torres and Rojas-Medar [17], [18], and Rojas-Medar and Boldrini [21], also studied the system (1.1)-(1.3) and stablished the existence and uniqueness of local strong solutions, global strong solutions, and existence and uniqueness of weak solutions, respectively, by using the spectral Galerkin method, reaching the same level of knowledge as in the case of the classical Navier-Stokes equations.

It has to be pointed out that similar time-dependent problems but for the Navier-Stokes equations have been studied by many different authors. This is the case, for instance, of the works by, J.L. Lions [9] (see also this book of J.L. Lions [10]), H. Fujita and N. Sauer [7], H. Morimoto [16], R. Salvi [22]. In particular, we would like to emphasize that the arguments in J.L. Lions [9], [10], requires Ω_t to be nondecreasing with respect to t (see problem 11.9, p. 426 of this book). Our paper, other that generalize these previous works in the sense that problem (1.1)-(1.3) includes the microrotational velocity and magnetic field, does not assume this nondecreasing condition on Ω_t.

This paper is organized as follows. After this brief introduction, in section 2, we introduce various functions spaces. Next, in section 3, we state the main theorem of existence of the weak solutions.

2. Function Spaces and Preliminaries

The functions in this paper are either $\mathbb{R}$ or $\mathbb{R}^3$-valued and we will not distinguish these two situations in our notations. To which case we refer to will be clear from the context. We denote $\|\cdot\|_{L^2}$ by $|\cdot|$.

Now, we give the precise definition of the time-dependent space domain Q where our initial boundary-value problems associated to the problem (1.1)-(1.3) has been formulated.

Let $T > 0$, we consider the function $R : [0,T] \longrightarrow \mathbb{R}^9$, that is, $R(t)$ is a 3×3 matrix. Let Ω be an open bounded set of $\mathbb{R}^3$, which, without loss of generality, can be considered containing the origin of $\mathbb{R}^3$.

We suppose that the boundary $\partial\Omega$ of Ω is smooth. We consider the sets

$$\Omega_t = \{x = yR(t)\,;y \in \Omega\},\ \ 0 \leq t \leq T. \tag{2.1}$$

It is worth noting that such domains Ω_t, $0 \leq t \leq T$, generate a non-cylindrical time-dependent domain of $\mathbb{R}^3 \times \mathbb{R}$, $Q = \bigcup_{0<t<T} \Omega_t \,\mathrm{x}\, \{t\}$ whose lateral boundary $\partial Q = \bigcup\limits_{0<t<T} \partial\Omega_t \,\mathrm{x}\, \{t\}$ is supposed regular.

We make the following hypothesis on $R(t)$: $R(t) = \sigma(t)\, M$, where $\sigma : [0,T] \longrightarrow \mathbb{R}, \sigma \in C^1([0,T])$, $\sigma(t) > 0$, M is a 3x3 matrix whose entries are real constant and that there exist its inverse.

The main goal in this paper is to show existence of weak solutions for the initial value problem (1.1)-(1.3). Our strategy for setting this question consists of transforming problem (1.1)-(1.3) into another initial-value problem in a cylindrical domain whose sections are not time-dependent. This is done by means of a suitable change of variable. Next, this new initial value problem is tretated using Galerkin's approximation and the Aubin-Lions Lemma. We conclude returning to Q using the inverse of the above change of variable.

Sets of type (2.1) where $R(t) = \sigma(t)\, I$, I identity $n \times n$-matrix, and Ω is the unit ball of $\mathbb{R}^n$ were considered by R. Del Passo and M. Ughi [4] to study a certain class of parabolic equations in noncylindrical domains.

Also, L. A. Medeiros and M. Milla-Miranda [13], [14] used the sets of type (2.1) where $R(t) = \sigma(t)\, I$, and Ω is a bounded open set of $\mathbb{R}^n$, with regular boundary $\partial\Omega$ and $0 \in \Omega$ and $\min \sigma(t) > 0$, to study exact controllability for Schröndinger equation in non-cylindrical domains.

C. Conca and Rojas-Medar [2] use the analogous domain that [4] to study the Boussinesq problem; M.A. Rojas-Medar and R. Beltrán-Barrios [20] for the magnetohydrodynamic type equations. The formulation of the general class of domains considered in this paper was given by M. Milla-Miranda and J. Límaco-Ferrel [15] to study the classical Navier-Stokes equations.

In order to state the main result we introduce some spaces, following the notation of [15], let $\mathcal{V}_t$ be the space $\mathcal{V}_t = \{\phi \in (C_0^\infty(\Omega_t))^3 \,/\, \operatorname{div}\phi = 0\}$ and $V_s(\Omega_t)$ be the closure of $\mathcal{V}_t$ in the space $(H^s(\Omega_t))^3$, $s \in \mathbb{R}_+$. We use the particular notation $V_1(\Omega_t) = V(\Omega_t)$ and $V_0(\Omega_t) = H(\Omega_t)$.

The inner product of $V(\Omega_t)$ and $H(\Omega_t)$ are

$$((u,v))_t = \sum_{i,j=1}^{3} \int_{\Omega_t} \frac{\partial u_i(x)}{\partial x_j}\frac{\partial v_i(x)}{\partial x_j} dx, \quad (u,v)_t = \sum_{i=1}^{3} \int_{\Omega_t} u_i(x) v_i(x) dx.$$

We observe that $V_s(\Omega_t) \hookrightarrow (H_0^1(\Omega_t))^3$ continuously for $s > \frac{1}{2}$ and

$$V(\Omega_t) = \{u \in (H_0^1(\Omega_t))^3 \,/\, \operatorname{div} u = 0\}$$

We introduce in similar way the spaces $V_s(\Omega)$, in this case $\mathcal{V}$ has the form

$$\mathcal{V} = \{\psi \in (C_0^\infty(\Omega))^3 \,/\, \operatorname{div}(\psi M^{-1}) = 0\}.$$

We put $V_1(\Omega) = V$, $V_0(\Omega) = H$ and

$$(u,v)_H = (u,v)_{L^2}, \ (u,v)_V = ((u,v))_{L^2} = (\nabla u, \nabla v)_{L^2}.$$

Also, $H^{-s}(\Omega)$ and $(V_s(\Omega))^*$ will denote the topological dual of $H^s(\Omega)$ and $V_s(\Omega)$ respectively.

In continuation, we will define the notion of weak solutions for the problem (1.1)-(1.3).

Definition. Let u_0, $b_0 \in H(\Omega_0)$ and $w_0 \in L^2(\Omega_0)$. We say that (u,w,b) is a weak solution of problem (1.1)-(1.3), if and only if u, $b \in L^2(0,T;V(\Omega_t)) \cap$

$L^\infty(0,T;H(\Omega_t))$, $w \in L^2(0,T;H_0^1(\Omega_t)) \cap L^\infty(0,T;L^2(\Omega_t))$, satisfying:

$$
\begin{gathered}
-\int_0^T (u,\varphi_t)_t dt + (\mu+\chi)\int_0^T (\nabla u, \nabla\varphi)_t dt + \int_0^T (u.\nabla u, \varphi)_t dt \\
-r\int_0^T (b.\nabla b, \varphi)_t dt = \chi \int_0^T (\text{rot } w, \varphi)_t dt + \int_0^T (f,\varphi)_t dt \\
-j\int_0^T (w,\phi_t)_t dt + \gamma\int_0^T (\nabla w, \nabla\phi)_t dt + \int_0^T (u.\nabla w, \phi)_t dt + 2\chi\int_0^T (w,\phi)_t dt \\
+(\alpha+\beta)\int_0^T (\text{div } w, \text{div } \phi)_t dt = \chi\int_0^T (\text{rot } u, \phi)_t dt + \int_0^T (g,\phi)_t dt \\
-\int_0^T (b,\psi_t)_t dt + \nu\int_0^T (\nabla b, \nabla\psi)_t dt + \int_0^T (u.\nabla b, \psi)_t dt - \int_0^T (b.\nabla u, \psi)_t dt = 0
\end{gathered}
$$

$\forall\ \varphi,\ \phi,\ \psi\ \in C^1(\bar{Q})$ *with compact support* $\subseteq Q$, $\text{div}\,\varphi = \text{div}\,\psi = 0$,

$u(0) = u_0, \quad w(0) = w_0, \quad b(0) = b_0.$

Remark. As it usual, the above regularity condition is enough to guarantee that the initial conditions has a meaning.

Our result is

Theorem 1. *Under the above hypotheses on* Ω_t*. If* $u_0, b_0 \in H(\Omega_0)$*,* $w_0 \in L^2(\Omega_0)$*, and* $f, g \in L^2(0,T;L^2(\Omega_t))$*, then there exists a weak solution* (u,w,b) *of (1.1)-(1.3). Therefore,*

$$u, b \in C_{\mathrm{w}}([0,T];H(\Omega_t)) \cap C([0,T];(V_{3/2}(\Omega_t))^*) \tag{2.2}$$

$$\text{and} \quad w \in C_{\mathrm{w}}([0,T];L^2(\Omega_t)) \cap C([0,T];H^{-3/2}(\Omega_t)). \tag{2.3}$$

Remark 1. In the proof of Theorem 1, the norm of a matrix will be denote by $\|\cdot\|$, since in finite-dimensional spaces all the norms are equivalent.

Also, C will be denote a generic positive constant that only depend up Ω, of fixed parameters $\mu, \chi, j, \nu, r, \gamma, \alpha, \beta$ and $\max\limits_{0\le t\le T}\{\|R(t)\|, \|R'(t)\|,\ \|R^{-1}(t)\|\}$.

3. Proof of Theorem 1

Let us introduce the transformation $\Phi : Q \to U$, given by $\Phi(x,t) = (xR^{-1}(t), t)$, where $U = \Omega \times (0,T)$. Since $\sigma(t)$ is a C^1-function, the transformation Φ is a C^1-diffeomorphism and its inverse $\Phi^{-1} : U \to Q$ satisfies $\Phi^{-1}(y,t) = (yR(t), t)$.

We also define

$$\begin{aligned} &v(y,t) = u(yR(t),t), \quad z(y,t) = w(yR(t),t), \quad h(y,t) = b(yR(t),t), \\ &q(y,t) = p(yR(t),t), \quad f_1(y,t) = f(yR(t),t), \quad g_1(y,t) = g(yR(t),t). \end{aligned} \tag{3.1}$$

We denote $R(t) = (\sigma_{ij}(t))$, $R^{-1}(t) = (\beta_{ij}(t))$ and $K(t) = (R^{-1}(t))^t$. Also, since $R(t)R^{-1}(t) = I$ we have

$$R(t)(R^{-1}(t))' = -R'(t)R^{-1}(t) \tag{3.2}$$

Consequently, using (3.1)-(3.2), we get

$$\begin{aligned} &u_t = -yR'(t)R^{-1}(t).\nabla v + v_t && u.\nabla u = vR^{-1}(t).\nabla v \\ &\triangle u = \sum_{i,l=1}^{3} \frac{\partial}{\partial y_i}\Big(\sum_{k=1}^{3} \beta_{kl}(t)\beta_{ki}(t)\frac{\partial v}{\partial y_l}\Big) && \nabla p = \nabla q K(t) \\ &\nabla(b.b) = \nabla(h.h)K(t) && \nabla \mathrm{div}\, w = \nabla \mathrm{div}(zR^{-1}(t))K(t) \\ &\mathrm{rot}\, w = \sum_{i=1}^{3} \nabla z_i A_i(t) \text{ onde } A_i(t) = K(t)K_{ii}(-1)K_{\alpha_i}(-1)K_{\alpha_i\gamma_i}, \text{ with} \\ &\alpha_i = (-1)^i + \Big(\frac{4+(2-i)(3-i)}{2}\Big), \ \gamma_i = (-1)^{i+1} + \Big(\frac{4+(2-i)(i-1)}{2}\Big) \end{aligned}$$

and $K_{ii}(-1)$, $K_{\alpha_i}(-1)$, $K_{\alpha_i\gamma_i}$ are elementary transformations of matrixes, $\mathrm{div} u = \mathrm{div}(vR^{-1}(t))$.

Therefore, the system (1.1)-(1.3) defined on Q is transformed on U into the system:

$$v_t - (\mu+\chi)\sum_{i,l=1}^{3}\frac{\partial}{\partial y_i}(\sum_{k=1}^{3}\beta_{kl}(t)\beta_{ki}(t)\frac{\partial v}{\partial y_l}) + vR^{-1}(t).\nabla v - yR'(t)R^{-1}(t).\nabla v$$

$$+\nabla(q+\frac{r}{2}h.h)K(t) = f_1 + rhR^{-1}(t).\nabla h + \chi\sum_{i=1}^{3}\nabla z_i A_i(t), \quad (3.3)$$

$$jz_t - \gamma\sum_{i,l=1}^{3}\frac{\partial}{\partial y_i}(\sum_{k=1}^{3}\beta_{kl}(t)\beta_{ki}(t)\frac{\partial z}{\partial y_l}) + jvR^{-1}(t).\nabla z - jyR'(t)R^{-1}(t).\nabla z$$

$$+2\chi z - (\alpha+\beta)\nabla \mathrm{div}(zR^{-1}(t))K(t) = g_1 + \chi\sum_{i=1}^{3}\nabla v_i A_i(t), \quad (3.4)$$

$$h_t - \nu\sum_{i,l=1}^{3}\frac{\partial}{\partial y_i}(\sum_{k=1}^{3}\beta_{kl}(t)\beta_{ki}(t)\frac{\partial h}{\partial y_l}) - yR'(t)R^{-1}(t).\nabla h + vR^{-1}(t).\nabla h$$

$$-hR^{-1}(t).\nabla v = 0, \quad (3.5)$$

$$\mathrm{div}(vM^{-1}) = 0 \text{ and } \mathrm{div}(hM^{-1}) = 0 \text{ in } U, \quad (3.6)$$

$$v(y,t) = z(y,t) = h(y,t) = 0 \text{ on } \partial\Omega\times(0,T), \quad (3.7)$$

$$v(y,0) = v_0(y),\ z(y,0) = z_0(y),\ h(y,0) = h_0(y) \text{ in } \Omega. \quad (3.8)$$

The notion of weak solution for (3.3)-(3.8) is completely similar to the ones for (1.1)-(1.3).

To prove the existence of solutions of the transformed system (3.3)-(3.8) we will use the spectral Galerkin method. That is, we fix $s = 3/2$ and we consider the Hilbert special basis $\{\varphi^i(y)\}_{i=}^{\infty}$ of $V_s(\Omega)$ and $\{\phi^i(y)\}_{i=1}^{\infty}$ of $H_0^s(\Omega)$, whose elements we will choose as the solutions of the following spectral problems:

$$(\varphi^i, v)_s = \lambda_i(\varphi^i, v),\ \forall\, v\in V_s(\Omega), \quad (\phi^i, w)_s = \tilde\lambda_i(\phi^i, w),\ \forall\, w\in H_0^s(\Omega).$$

Let V^k be the subspace of $V_s(\Omega)$ spanned by $\{\varphi^1(y),\dots,\varphi^k(y)\}$ and H_k be the subspace of $H_0^s(\Omega)$ spanned by $\{\phi^1(y),\dots,\phi^k(y)\}$, respectively. For every $k\geq 1$, we define approximations v^k, z^k and h^k of v, z and h respectively, by means of the following finite expansions:

$$v^k(y,t) = \sum_{i=1}^{k} c_{ik}(t)\varphi^i(y), \quad z^k(y,t) = \sum_{i=1}^{k} d_{ik}(t)\phi^i(y), \quad h^k(y,t) = \sum_{i=1}^{k} e_{ik}(t)\varphi^i(y)$$

for $t\in(0,T)$, where the coefficients (c_{ik}), (d_{ik}) and (e_{ik}) will be calculated in such way that v^k, z^k and h^k solve the following approximations of system (3.3)-(3.8):

$$(v_t^k, \varphi) + (\mu + \chi)\tilde{a}(t; v^k, \varphi) + \tilde{b}(t; v^k, v^k, \varphi) - \tilde{c}(t; v^k, \varphi) = r\tilde{b}(t; h^k, h^k, \varphi)$$
$$+(f_1, \varphi) + \chi(\sum_{i=1}^{3} \nabla z_i^k A_i(t), \varphi), \tag{3.9}$$
$$j(z_t^k, \phi) + \gamma\tilde{a}(t; z^k, \phi) + (\alpha + \beta)(\operatorname{div}(z^k R^{-1}(t)), \operatorname{div}(\phi R^{-1}(t)))$$
$$+j\tilde{b}(t; v^k, z^k, \phi) - j\tilde{c}(t; z^k, \phi) + 2\chi(z^k, \phi)$$
$$= (g_1, \phi) + \chi(\sum_{i=1}^{3} \nabla v_i^k A_i(t), \phi), \tag{3.10}$$
$$(h_t^k, \psi) + \nu\tilde{a}(t; h^k, \psi) - \tilde{c}(t; h^k, \psi) + \tilde{b}(t; v^k, h^k, \psi) = \tilde{b}(t; h^k, v^k, \psi), \tag{3.11}$$
$\forall\ \varphi,\ \psi \in V^k$ and $\forall\ \phi \in H_k$,
$$v^k(0) = v_0^k,\ z^k(0) = z_0^k,\ h^k(0) = h_0^k, \tag{3.12}$$

where $v_0^k \longrightarrow v_0,\ h_0^k \longrightarrow h_0$ in $H(\Omega)$ and $z_0^k \longrightarrow z_0$ in $L^2(\Omega)$ as $k \longrightarrow \infty$ and

$$\tilde{a}(t; u, w) = \sum_{j=1}^{3} \int_\Omega \sum_{i,l=1}^{3} (\sum_{k=1}^{3} \beta_{kl}(t)\beta_{ki}(t)) \frac{\partial u_j}{\partial y_l} \frac{\partial w_j}{\partial y_i} dy$$
$$\tilde{b}(t; u, v, w) = \sum_{j=1}^{3} \int_\Omega \sum_{i,l=1}^{3} \beta_{il}(t) u_i \frac{\partial v_j}{\partial y_l} w_j dy$$
$$\tilde{c}(t; u, w) = \sum_{j=1}^{3} \int_\Omega \sum_{i,l,k=1}^{3} \sigma'_{ki}(t)\beta_{il}(t) y_k \frac{\partial u_j}{\partial y_l} w_j dy$$

for vector-valued functions u, w, v for which the integrals are well defined.

We observe that the following identity was used

$$(\nabla(q + \frac{r}{2}h.h)K(t), \varphi) = -(q + \frac{r}{2}h.h, \operatorname{div}\varphi R^{-1}(t)) = 0, \quad \forall \varphi \in V^k.$$

Equations (3.9)-(3.12) is a system of ordinary differential equations for the coefficients functions $c_{ik}(t), d_{ik}(t)$ and $e_{ik}(t)$, which defines v^k, z^k and h^k in an interval $[0, t_k[$. We will show some a priori estimates independent of k and t, in order to take $t_k = T$. Also, we will prove that (v^k, z^k, h^k)converges in appropriate sense to a solution (u, z, h) of (3.3)(-(3.8).

We prove the following lemma.

Lemma 1. *The transformed system (3.3)-(3.8) admits at least one weak solution* (v, z, h) *satisfying the following:*

$$
\begin{aligned}
&v, h \in L^2(0,T;V(\Omega)) \cap L^\infty(0,T;H(\Omega)),\\
&z \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)).
\end{aligned}
$$

Proof. Setting $\varphi = v^k, \phi = z^k$ and $\psi = rh^k$ in (3.9)-(3.11) and observing that $\tilde{b}(t;u,v,v) = 0$, we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}|v^k|^2 + (\mu+\chi)|\nabla v^k K(t)|^2 &= (f_1, v^k) + \tilde{c}(t;v^k,v^k) + r\tilde{b}(t;h^k,h^k,v^k)\\
&\quad + \chi\Big(\sum_{i=1}^{3} \nabla z_i^k A_i(t), v^k\Big),\\
\frac{j}{2}\frac{d}{dt}|z^k|^2 + \gamma|\nabla z^k K(t)|^2 + 2\chi|z^k|^2 + (\alpha+\beta)|\mathrm{div}(z^k R^{-1}(t))|^2 &= (g_1, z^k)\\
&\quad + j\tilde{c}(t;z^k,z^k) + \chi\Big(\sum_{i=1}^{3} \nabla v_i^k A_i(t), z^k\Big),\\
\frac{r}{2}\frac{d}{dt}|h^k|^2 + r\nu|\nabla h^k K(t)|^2 &= r\tilde{c}(t;h^k,h^k) + r\tilde{b}(t;h^k,v^k,h^k).
\end{aligned}
$$

Adding the above equalities and observing that $\tilde{b}(t;u,v,w) + \tilde{b}(t;u,w,v) = 0$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(|v^k|^2 + j|z^k|^2 + r|h^k|^2) + (\mu+\chi)|\nabla v^k K(t)|^2 + \gamma|\nabla z^k K(t)|^2\\
&\quad + r\nu|\nabla h^k K(t)|^2 + 2\chi|z^k|^2 + (\alpha+\beta)|\mathrm{div}(z^k R^{-1}(t))|^2\\
&= (f_1, v^k) + (g_1, z^k) + \tilde{c}(t;v^k,v^k) + j\,\tilde{c}(t;z^k,z^k) + r\,\tilde{c}(t;h^k,h^k)\\
&\quad + \chi\Big(\sum_{i=1}^{3} \nabla z_i^k A_i(t), v^k\Big) + \chi\Big(\sum_{i=1}^{3} \nabla v_i^k A_i(t), z^k\Big). \qquad (3.13)
\end{aligned}
$$

Now we will estimate the right-hand side of (3.13). By using the Hölder and

Young inequalities, we obtain

$$
\begin{aligned}
&|(f_1, v^k)| \le |f_1||v^k| \le \frac{1}{2}|f_1|^2 + \frac{1}{2}|v^k|^2,\\
&|(g_1, z^k)| \le |g_1||z^k| \le \frac{1}{4\chi}|g_1|^2 + 2\chi|z^k|^2,\\
&|\tilde{c}(t; v^k, v^k)| \le \frac{\mu+\chi}{4}|\nabla v^k K(t)|^2 + (\frac{\|R'(t)\|^2\|R^{-1}(t)\|^2\|R(t)\|^2}{\mu+\chi}\|y\|^2_{L^\infty})|v^k|^2,\\
&|j\tilde{c}(t; z^k, z^k)| \le \frac{\gamma}{4}|\nabla z^k K(t)|^2 + (\frac{j\|R'(t)\|^2\|R^{-1}(t)\|^2\|R(t)\|^2}{\gamma}\|y\|^2_{L^\infty})j|z^k|^2,\\
&|r\,\tilde{c}(t; h^k, h^k)| \le \frac{r\,\nu}{2}|\nabla h^k K(t)|^2 + (\frac{\|R'(t)\|^2\|R^{-1}(t)\|^2\|R(t)\|^2}{2\nu}\|y\|^2_{L^\infty})r|h^k|^2,\\
&|\chi(\sum_{i=1}^{3}\nabla z_i^k A_i(t), v^k)| \le \frac{\gamma}{4}|\nabla z^k K(t)|^2 + \frac{\chi^2}{\gamma}|v^k|^2,\\
&|\chi(\sum_{i=1}^{3}\nabla v_i^k A_i(t), z^k)| \le \frac{\mu+\chi}{4}|\nabla v^k K(t)|^2 + (\frac{\chi^2}{j(\mu+\chi)})j|z^k|^2,
\end{aligned}
$$

whence, we arrive to the inequality

$$
\begin{aligned}
\frac{d}{dt}(|v^k|^2 + j|z^k|^2 + r|h^k|^2) + (\mu+\chi)|\nabla v^k K(t)|^2 + \gamma|\nabla z^k K(t)|^2\\
+r\nu|\nabla h^k K(t)|^2 + 2(\alpha+\beta)|\mathrm{div}(z^k R^{-1}(t))|^2\\
\le C(|f_1|^2 + |g_1|^2) + C(|v^k|^2 + j|z^k|^2 + r|h^k|^2), \qquad (3.14)
\end{aligned}
$$

where C is a positive constant that depends only of χ, μ, γ, j, $\max_{0\le t\le T}\|R'(t)\|$, $\max_{0\le t\le T}\|R^{-1}(t)\|$, $\max_{0\le t\le T}\|R(t)\|$ e $\|y\|_{L^\infty}$.

By integrating (3.14) from 0 to t, with $0 \le t \le T$, we conclude

$$
\begin{aligned}
&(|v^k(t)|^2 + j|z^k(t)|^2 + r|h^k(t)|^2) + (\mu+\chi)\int_0^t |\nabla v^k(s)K(s)|^2 ds\\
&+\gamma\int_0^t |\nabla z^k(s)K(s)|^2 ds + r\nu\int_0^t |\nabla h^k(s)K(s)|^2 ds\\
&\quad \le C_1\int_0^t (|f_1(s)|^2 + |g_1(s)|^2)ds + C\int_0^t (|v^k(s)|^2 + j|z^k(s)|^2 + r|h^k(s)|^2)ds\\
&\qquad +|v^k(0)|^2 + j|z^k(0)|^2 + r|h^k(0)|^2.
\end{aligned}
$$

Due to the choice of v_0^k, z_0^k and h_0^k, there exists C_2 independent of k such that $|v_0^k| \le C_2|v_0|$, $|z_0^k| \le C_2|z_0|$ and $|h_0^k| \le C_2|h_0|$.

Then, since $f_1,\ g_1 \in L^2(0,T;L^2(\Omega))$, result

$$\begin{aligned}(|v^k(t)|^2 + j|z^k(t)|^2 + r|h^k(t)|^2) + (\mu+\chi)\int_0^t |\nabla v^k(s)K(s)|^2 ds \\ +\gamma\int_0^t |\nabla z^k(s)K(s)|^2 ds + r\nu\int_0^t |\nabla h^k(s)K(s)|^2 ds \\ \le C_3 + C\int_0^t (|v^k(s)|^2 + j|z^k(s)|^2 + r|h^k(s)|^2) ds.\end{aligned}$$

By using Gronwall's inequality, we have

$$\begin{aligned}(|v^k(t)|^2 + j|z^k(t)|^2 + r|h^k(t)|^2) + (\mu+\chi)\int_0^t |\nabla v^k(s)K(s)|^2 ds \\ +\gamma\int_0^t |\nabla z^k(s)K(s)|^2 ds + r\nu\int_0^t |\nabla h^k(s)K(s)|^2 ds \le C.\end{aligned}$$

Thus for all k, we have that v^k, z^k and h^k exist globally in t. Now, we put $N = \max_{0\le t\le T}\|R(t)\|$, then we observe that $\dfrac{1}{N^2}|\nabla v^k|^2 \le \dfrac{1}{\|R(t)\|^2}|\nabla v^k|^2 \le |\nabla v^k K(t)|^2$, whence $\displaystyle\int_0^t |\nabla v^k(s)|^2 ds \le N^2 C$. Moreover,

$$\begin{aligned}&(v^k),\ (h^k) \text{ are bounded in } L^\infty(0,T;H(\Omega)) \cap L^2(0,T;V(\Omega)) \\ &\text{and} \quad (z^k) \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)).\end{aligned} \tag{3.15}$$

The next step of the proof consists of proving that $(v_t^k), (h_t^k)$ are bounded in $L^2(0,T;(V_{3/2}(\Omega))^*)$ and that (z_t^k) is bounded in $L^2(0,T;H^{-3/2}(\Omega))$.

We consider $P_k : H(\Omega) \longrightarrow V^k$ and $R_k : L^2(\Omega) \longrightarrow H_k$, defined by

$$P_k u = \sum_{i=1}^k (u,\varphi^i)\varphi^i \quad \text{and} \quad R_k w = \sum_{i=1}^k (w,\phi^i)\phi^i.$$

Since $V_s(\Omega) \hookrightarrow H(\Omega)$ and $H_0^s \hookrightarrow L^2(\Omega)$; $V^k \hookrightarrow V_s(\Omega)$ and $H_k \hookrightarrow H_0^s(\Omega)$ we can consider $P_k : V_s(\Omega) \longrightarrow V_s(\Omega)$ and $R_k : H_0^s(\Omega) \longrightarrow H_0^s(\Omega)$. It is easily to see that $P_k \in L(V_s(\Omega), V_s(\Omega))$ and $R_k \in L(H_0^s(\Omega), H_0^s(\Omega))(L(X,Y)$ denote the space of the bounded operators of X into Y), hence $P_k^* : (V_s(\Omega))^* \longrightarrow (V_s(\Omega))^*$ and $R_k^* : H^{-s}(\Omega) \longrightarrow H^{-s}(\Omega)$, defined by $\langle P_k^*(v), \omega\rangle = \langle v, P_k(\omega)\rangle$ lies in $L((V_s(\Omega))^*,\ (V_s(\Omega))^*)$ and $\|P_k^*\| \le \|P_k\| \le 1$. Analogously, for R_k^*. We also observe that the autofunctions φ^i and ϕ^i are invariants by P_k and R_k, respectively.

From it and (3.9)-(3.11) $\forall\,\omega,\,\eta\in V^k$ and $\forall\,\xi\in H_k$, we have

$$
\begin{aligned}
(v_t^k,\omega)=\langle P_k^*(\,(\mu+\chi)(\sum_{i,l=1}^{3}\frac{\partial}{\partial y_i}(\sum_{k=1}^{3}\beta_{kl}(t)\beta_{ki}(t)\frac{\partial v^k}{\partial y_l}))-v^kR^{-1}(t).\nabla v^k\\
+yR'(t)R^{-1}(t).\nabla v^k+f_1+rh^kR^{-1}(t).\nabla h^k+\chi\sum_{i=1}^{3}\nabla z_i^kA_i(t)\,),\omega\rangle,\\
j(z_t^k,\xi)=\langle R_k^*(\,\gamma(\sum_{i,l=1}^{3}\frac{\partial}{\partial y_i}(\sum_{k=1}^{3}\beta_{kl}(t)\beta_{ki}(t)\frac{\partial z^k}{\partial y_l}))-jv^kR^{-1}(t).\nabla z^k\\
+jyR'(t)R^{-1}(t).\nabla z^k-2\chi z^k+(\alpha+\beta)\nabla\mathrm{div}(z^kR^{-1}(t))K(t)\\
+g_1+\chi\sum_{i=1}^{3}\nabla v_i^kA_i(t)\,),\xi\rangle,\\
(h_t^k,\eta)=\langle P_k^*(\,\nu(\sum_{i,l=1}^{3}\frac{\partial}{\partial y_i}(\sum_{k=1}^{3}\beta_{kl}(t)\beta_{ki}(t)\frac{\partial h^k}{\partial y_l}))+yR'(t)R^{-1}(t).\nabla h^k\\
-v^kR^{-1}(t).\nabla h^k+h^kR^{-1}(t).\nabla v^k\,),\eta\rangle.
\end{aligned}
$$

Hence, by taking $\omega=P_ku$, $\eta=P_kb$, for $u,b\in V_s(\Omega)$ and $\xi=R_kw$ for $w\in H_0^s(\Omega)$, we obtain

$$
\begin{aligned}
(v_t^k,u)=\langle P_k^*(\,(\mu+\chi)\sum_{i,l=1}^{3}\frac{\partial}{\partial y_i}(\sum_{k=1}^{3}\beta_{kl}(t)\beta_{ki}(t)\frac{\partial v^k}{\partial y_l}))-P_k^*(v^kR^{-1}(t).\nabla v^k)\\
+P_k^*(yR'(t)R^{-1}(t).\nabla v^k)+P_k^*(f_1)+P_k^*(rh^kR^{-1}(t).\nabla h^k)\\
+P_k^*(\chi\sum_{i=1}^{3}\nabla z_i^kA_i(t)),u\rangle, \qquad (3.16)\\
j(z_t^k,w)=\langle R_k^*(\gamma\sum_{i,l=1}^{3}\frac{\partial}{\partial y_i}(\sum_{k=1}^{3}\beta_{kl}(t)\beta_{ki}(t)\frac{\partial z^k}{\partial y_l}))-R_k^*(jv^kR^{-1}(t).\nabla z^k)\\
+R_k^*(jyR'(t)R^{-1}(t).\nabla z^k)-R_k^*(2\chi z^k)+R_k^*(\chi\sum_{i=1}^{3}\nabla v_i^kA_i(t))\\
+R_k^*(g_1)+R_k^*((\alpha+\beta)\nabla\mathrm{div}(z^kR^{-1}(t))K(t)),w\rangle, \qquad (3.17)\\
(h_t^k,b)=\langle P_k^*(\nu\sum_{i,l=1}^{3}\frac{\partial}{\partial y_i}(\sum_{k=1}^{3}\beta_{kl}(t)\beta_{ki}(t)\frac{\partial h^k}{\partial y_l}))+P_k^*(yR'(t)R^{-1}(t).\nabla h^k)\\
-P_k^*(v^kR^{-1}(t).\nabla h^k)+P_k^*(h^kR^{-1}(t).\nabla v^k),b\rangle. \qquad (3.18)
\end{aligned}
$$

We observe that

$$\begin{aligned}
&\|P_k^*(\,(\mu+\chi)\sum_{i,l=1}^{3}\frac{\partial}{\partial y_i}(\sum_{k=1}^{3}\beta_{kl}(t)\beta_{ki}(t)\frac{\partial v^k}{\partial y_l}))\|_{(V_s)^*}\\
&\quad\le(\mu+\chi)\sup_{\|u\|_{V_s}\le 1}|(\,\nabla v^kK(t),\nabla uK(t)\,)|\\
&\quad\le C(\mu+\chi)\max_{0\le t\le T}\{\|R^{-1}(t)\|^2\}\sup_{\|u\|_{V_s}\le 1}|\nabla v^k||\nabla u|\le C|\nabla v^k|,
\end{aligned}$$

them, from (3.15), we have

$$\int_0^t\|P_k^*(\,(\mu+\chi)\sum_{i,l=1}^{3}\frac{\partial}{\partial y_i}(\sum_{k=1}^{3}\beta_{kl}(s)\beta_{ki}(s)\frac{\partial v^k}{\partial y_l}(s)))\|_{(V_s)^*}^2ds\le C. \tag{3.19}$$

Analogously,

$$\begin{aligned}
\|P_k^*(yR'(t)R^{-1}(t).\nabla v^k)\|_{(V_s)^*} &\le \sup_{\|u\|_{V_s}\le 1}|<yR'(t)R^{-1}(t).\nabla v^k,u>|\\
&\le \sup_{\|u\|_{V_s}\le 1}\|R'(t)\|\|R^{-1}(t)\|\|y\|_\infty|\nabla v^k||u|\\
&\le C\max_{0\le t\le T}\{\|R'(t)\|\|R^{-1}(t)\|\}\|y\|_\infty|\nabla v^k|,
\end{aligned}$$

then

$$\int_0^t\|P_k^*(yR'(s)R^{-1}(s).\nabla v^k(s))\|_{(V_s)^*}^2ds\le C\int_0^t|\nabla v^k(s)|^2ds\le C. \tag{3.20}$$

Also,

$$\int_0^t\|P_k^*f_1(s)\|_{(V_s)^*}^2ds\le C\int_0^t|f_1(s)|^2ds\le C. \tag{3.21}$$

Observing that

$$\begin{aligned}
\|P_k^*(\chi\sum_{i=1}^{3}\nabla z_i^kA_i(t))\|_{(V_s)^*} &\le \sup_{\|u\|_{V_s}\le 1}|<\chi\sum_{i=1}^{3}\nabla z_i^kA_i(t),u>|\\
&\le C\|R^{-1}(t)\||\nabla z^k|\le C|\nabla z^k|,
\end{aligned}$$

and (3.15), we have

$$\int_0^t\|P_k^*(\chi\sum_{i=1}^{3}\nabla z_i^k(s)A_i(s))\|_{(V_s)^*}^2ds\le C\int_0^t|\nabla z^k(s)|^2ds\le C. \tag{3.22}$$

Now, to estimate the term $P_k^*(v^k R^{-1}(t).\nabla v^k)$, we will use the following interpolation result whose proof can be found in Lions [10, p. 73]:

Lemma 2. *If (u^k) is bounded in $L^2(0,T;V(\Omega)) \cap L^\infty(0,T;H(\Omega))$, then (u^k) is also bounded in $L^4(0,T;L^p(\Omega))$, where $\frac{1}{p} = \frac{1}{2} - \frac{1}{2n}$.*

Using the Sobolev imbedding $H^{s-1} \hookrightarrow L^3$ $(s = 3/2)$, we have

$$\begin{aligned}
\|P_k^*(v^k R^{-1}(t).\nabla v^k)\|_{(V_s)^*} &\leq \sup_{\|u\|_{V_s}\leq 1} |< v^k R^{-1}(t).\nabla v^k, u >| \\
&\leq \sup_{\|u\|_{V_s}\leq 1} \|v^k R^{-1}(t)\|_{L^3} \|\nabla u\|_{L^3} \|v^k\|_{L^3} \\
&\leq C\|R^{-1}(t)\|\|v^k\|_{L^3}^2 \sup_{\|u\|_{V_s}\leq 1} \|\nabla u\|_{H^{s-1}} \\
&\leq C\|R^{-1}(t)\|\|v^k\|_{L^3}^2 \sup_{\|u\|_{V_s}\leq 1} \|u\|_{H^s} \\
&\leq C\|R^{-1}(t)\|\|v^k\|_{L^3}^2 \leq C\|v^k\|_{L^3}^2,
\end{aligned}$$

and from (3.15) using the Lemma 2 $(n = 3)$, we have that (v^k) is bounded in $L^4(0,T;L^3(\Omega))$. Moreover, we get

$$\int_0^t \|P_k^*(v^k(s)R^{-1}(s).\nabla v^k(s))\|_{(V_s)^*}^2 ds \leq C \int_0^t \|v^k(s)\|_{L^3}^4 ds \leq C. \tag{3.23}$$

Analogously,

$$\int_0^t \|P_k^*(rh^k(s)R^{-1}(s).\nabla h^k(s))\|_{(V_s)^*}^2 ds \leq C \int_0^t \|h^k(s)\|_{L^3}^4 ds \leq C. \tag{3.24}$$

By using the estimates (3.19)-(3.24) in (3.16), we get

$$\int_0^t \|v_t^k(s)\|_{(V_s)^*}^2 ds \leq C.$$

Therefore, (v_t^k) is bounded in $L^2(0,T;(V_s(\Omega))^*)$. Analogously we can proved that (h_t^k) is bounded in $L^2(0,T;(V_s(\Omega))^*)$

From (3.17), we have

$$
\begin{aligned}
j\|z_t^k\|_{H^{-s}} \leq & \|R_k^*(\gamma \sum_{i,l=1}^{3} \frac{\partial}{\partial y_i}(\sum_{k=1}^{3} \beta_{kl}(t)\beta_{ki}(t)\frac{\partial z^k}{\partial y_l}))\|_{H^{-s}} + \|R_k^*(g_1)\|_{H^{-s}} \\
& +\|R_k^*(jyR'(t)R^{-1}(t).\nabla z^k)\|_{H^{-s}} + \|R_k^*(jv^k R^{-1}(t).\nabla z^k)\|_{H^{-s}} \\
& +\|R_k^*(2\chi z^k)\|_{H^{-s}} + \|R_k^*(\chi \sum_{i=1}^{3} \nabla v_i^k A_i(t))\|_{H^{-s}} \\
& +\|R_k^*((\alpha+\beta)\nabla \mathrm{div}(z^k R^{-1}(t))K(t))\|_{H^{-s}}. \qquad (3.25)
\end{aligned}
$$

We only estimate the last term of (3.25), the others terms are analogously estimate. We have

$$
\begin{aligned}
&\|R_k^*((\alpha+\beta)\nabla \mathrm{div}(z^k R^{-1}(t))K(t))\|_{H^{-s}} \\
&\quad \leq C \sup_{\|w\|_{H^s}\leq 1} |<\nabla \mathrm{div}(z^k R^{-1}(t))K(t), w>| \\
&\quad \leq C \sup_{\|w\|_{H^s}\leq 1} |(\,\mathrm{div}(z^k R^{-1}(t)), \mathrm{div}(wR^{-1}(t)\,)| \\
&\quad \leq C \sup_{\|w\|_{H^s}\leq 1} |\nabla (z^k R^{-1}(t))||\nabla (wR^{-1}(t))| \\
&\quad \leq C\|R^{-1}(t)\|^2 |\nabla z^k| \sup_{\|w\|_{H^s}\leq 1} \|w\|_{H^1} \leq C|\nabla z^k|,
\end{aligned}
$$

and from (3.15), we obtain

$$
\int_0^t \|R_k^*((\alpha+\beta)\nabla \mathrm{div}(z^k(s)R^{-1}(s))K(s))\|_{H^{-s}}^2 ds \leq C. \qquad (3.26)
$$

Therefore, (z_t^k) is bounded in $L^2(0,T;H^{-s}(\Omega))$.

Arguing as in the book of Lions [10, p. 76] and making use of the Aubin-Lions Lemma [10, p. 58], with $B_0 = V(\Omega), p_0 = 2,\ B = H(\Omega),\ B_1 = (V_s(\Omega))^*$ and $p_1 = 2$, we can conclude that there exists $v, h \in L^2(0,T;V(\Omega))$ such that, up to a subsequence which we shall denote again by the suffix k, there hold

$$
\begin{aligned}
& v^k \longrightarrow v \ \text{ and } \ h^k \longrightarrow h \ \text{ weak in } \ L^2(0,T;V(\Omega)), \\
& v^k \longrightarrow v \ \text{ and } \ h^k \longrightarrow h \ \text{ weak} -* \text{ in } \ L^\infty(0,T;H(\Omega)), \\
& v_t^k \longrightarrow v_t \ \text{ and } \ h_t^k \longrightarrow h_t \ \text{ weak in } \ L^2(0,T;(V_s(\Omega))^*), \\
& v^k \longrightarrow v \ \text{ and } \ h^k \longrightarrow h \ \text{ strong in } \ L^2(0,T;H(\Omega)),
\end{aligned}
$$

also with $B_0 = H_0^1(\Omega)$, $p_0 = 2$, $B_1 = H^{-s}(\Omega)$, $p_1 = 2$ and $B = L^2(\Omega)$, we have that there exist $z \in L^2(0,T;H_0^1(\Omega))$ such that

$$\begin{aligned}
z^k &\longrightarrow z \ \text{ weak in } \ L^2(0,T;H_0^1(\Omega)),\\
z^k &\longrightarrow z \ \text{ weak} -* \text{ in } \ L^\infty(0,T;L^2(\Omega)),\\
z_t^k &\longrightarrow z_t \ \text{ weak in } \ L^2(0,T;H^{-s}(\Omega)),\\
z^k &\longrightarrow z \ \text{ strong in } \ L^2(0,T;L^2(\Omega)).
\end{aligned}$$

Now, the next step is to take the limit. But, once the above convergence results, have been established, this is standard procedure and it follows the same patter as in Lions [10, p. 76]. Consequently, we obtain that (v, z, h) is a weak solution of problem (3.3)-(3.8), satisfying

$$\begin{aligned}
&(v_t,\varphi) + (\mu+\chi)(\nabla v K(t), \nabla\varphi K(t)) + (vR^{-1}(t).\nabla v, \varphi)\\
&\qquad -(yR'(t)R^{-1}(t).\nabla v,\varphi) = (f_1,\varphi) + r(hR^{-1}(t).\nabla h,\varphi)\\
&\qquad +\chi(\sum_{i=1}^{3}\nabla z_i A_i(t),\varphi), \qquad (3.27)\\
&j(z_t,\phi) + \gamma(\nabla z K(t),\nabla\phi K(t)) + j(vR^{-1}(t).\nabla z,\phi) + 2\chi(z,\phi)\\
&\qquad -j(yR'(t)R^{-1}(t).\nabla z,\phi) + (\alpha+\beta)(\mathrm{div}\,(zR^{-1}(t)), \mathrm{div}\,(\phi R^{-1}(t)))\\
&\qquad = (g_1,\phi) + \chi(\sum_{i=1}^{3}\nabla v_i A_i(t),\phi), \qquad (3.28)\\
&(h_t,\psi) + \nu(\nabla h K(t),\nabla\psi K(t)) - (yR'(t)R^{-1}(t).\nabla h,\psi) + (vR^{-1}(t).\nabla h,\psi)\\
&\qquad -(hR^{-1}(t).\nabla v,\psi) = 0, \qquad (3.29)\\
&\forall\,\varphi,\,\psi \in V(\Omega) \text{ and } \forall\,\phi \in H_0^1(\Omega),\\
&v(0) = v_0,\ z(0) = z_0,\ h(0) = h_0, \qquad (3.30)
\end{aligned}$$

in the distributional sense in $(0,T)$. This complete the proof of lemma.

□

To prove the theorem, we observe that the weak solution (v, z, h) of transformed problem (3.3)-(3.8), satisfies

$$-\int_0^T (v,\tilde{\varphi}_t)dt + (\mu+\chi)\int_0^T \tilde{a}(t;v,\tilde{\varphi})dt + \int_0^T \tilde{b}(t;v,v,\tilde{\varphi})dt - \int_0^T \tilde{c}(t;v,\tilde{\varphi})dt$$
$$= \int_0^T (f_1,\tilde{\varphi})dt + r\int_0^T \tilde{b}(t;h,h,\tilde{\varphi})dt + \chi\int_0^T (\sum_{i=1}^3 \nabla z_i A_i(t),\tilde{\varphi})dt \quad (3.31)$$

$$-j\int_0^T (z,\tilde{\phi}_t)dt + \gamma\int_0^T \tilde{a}(t;z,\tilde{\phi})dt + j\int_0^T \tilde{b}(t;v,z,\tilde{\phi})dt - j\int_0^T \tilde{c}(t;z,\tilde{\phi})dt$$
$$+2\chi\int_0^T (z,\tilde{\phi})dt + (\alpha+\beta)\int_0^T (\mathrm{div}\,(zR^{-1}(t)),\mathrm{div}\,(\tilde{\phi}R^{-1}(t)))dt$$
$$= \int_0^T (g_1,\tilde{\phi})dt + \chi\int_0^T (\sum_{i=1}^3 \nabla v_i A_i(t),\tilde{\phi})dt \quad (3.32)$$

$$-\int_0^T (h,\tilde{\psi}_t)dt + \nu\int_0^T \tilde{a}(t;h,\tilde{\psi})dt - \int_0^T \tilde{c}(t;h,\tilde{\psi})dt + \int_0^T \tilde{b}(t;v,h,\tilde{\psi})dt$$
$$-\int_0^T \tilde{b}(t;h,v,\tilde{\psi})dt = 0 \quad (3.33)$$

$\forall\,\tilde{\varphi},\ \tilde{\psi},\ \tilde{\phi} \in C^1(\bar{U})$ with compact support $\subseteq U$,

$\mathrm{div}(\tilde{\varphi}M^{-1}) = \mathrm{div}(\tilde{\psi}M^{-1}) = 0.$

To conclude the proof of theorem, let us consider a tests functions $\varphi,\ \phi,\ \psi \in C^1(\bar{Q})$ with compact supports Q such that $\mathrm{div}\,\varphi = 0,\ \mathrm{div}\,\psi = 0$ and define

$$\tilde{\varphi}(y,t) = det\,R(t)\,\varphi(yR(t),t),$$
$$\tilde{\phi}(y,t) = det\,R(t)\,\phi(yR(t),t),$$
$$\tilde{\psi}(y,t) = det\,R(t)\,\psi(yR(t),t).$$

It is easily seen that $\tilde{\varphi},\ \tilde{\psi},\ \tilde{\phi} \in C^1(\bar{U})$, with compact supports in U and $\mathrm{div}\,(\tilde{\varphi}M^{-1}) = \mathrm{div}\,(\tilde{\psi}M^{-1}) = 0$.

Integrating by parts,

$$-\int_0^T (v,\tilde{\varphi}_t)dt - \int_0^T \tilde{c}(t;v,\tilde{\varphi})dt = -\int_0^T det\,R(t)\,(v,\varphi_t)dt,$$

$$\int_0^T \tilde{a}(t;v,\tilde{\varphi})dt = \sum_{j=1}^3 \int_0^T \int_\Omega det\,R(t)\sum_{k,l=1}^3 \beta_{kl}(t)\frac{\partial v_j}{\partial y_l}\frac{\partial \varphi_j}{\partial x_k}dydt,$$

$$\int_0^T \tilde{b}(t;v,v,\tilde{\varphi})dt = -\sum_{k,j=1}^3 \int_0^T \int_\Omega det\,R(t)\,(v_k\frac{\partial \varphi_j}{\partial x_k}v_j)dydt,$$

$$\int_0^T (\mathrm{div}(zR^{-1}(t)),\mathrm{div}\,(\tilde{\phi}R^{-1}(t)))dt = \int_0^T det\,R(t)\,(\sum_{k,l=1}^3 \beta_{kl}(t)\frac{\partial z_k}{\partial y_l},\ \mathrm{div}\phi)dt,$$

where x_k is the k^{th} coordinate of $yR(t)$. By using the above identities in (3.31)-(3.33), we obtain

$$
\begin{aligned}
&-\int_0^T det\,R(t)\,(v,\varphi_t)dt + (\mu+\chi)\sum_{j=1}^{3}\int_0^T\int_\Omega det\,R(t)\,\sum_{k,l=1}^{3}\beta_{kl}(t)\frac{\partial v_j}{\partial y_l}\frac{\partial \varphi_j}{\partial x_k}dydt \\
&\quad -\sum_{k,j=1}^{3}\int_0^T\int_\Omega det\,R(t)\,(v_k\frac{\partial \varphi_j}{\partial x_k}v_j)dydt = \int_0^T det\,R(t)\,(f_1,\varphi)dt \\
&\quad\quad -r\sum_{k,j=1}^{3}\int_0^T\int_\Omega det\,R(t)\,(h_k\frac{\partial \varphi_j}{\partial x_k}h_j)dydt \\
&\quad\quad\quad +\chi\int_0^T det\,R(t)\,(\sum_{i=1}^{3}\nabla z_i A_i(t),\varphi)dt, \qquad (3.34)
\end{aligned}
$$

$$
\begin{aligned}
&-j\int_0^T det\,R(t)\,(z,\phi_t)dt + \gamma\sum_{j=1}^{3}\int_0^T\int_\Omega det\,R(t)\,\sum_{k,l=1}^{3}\beta_{kl}(t)\frac{\partial z_j}{\partial y_l}\frac{\partial \phi_j}{\partial x_k}dydt \\
&\quad -j\sum_{k,j=1}^{3}\int_0^T\int_\Omega det\,R(t)\,(v_k\frac{\partial \phi_j}{\partial x_k}z_j)dydt + 2\chi\int_0^T det\,R(t)\,(z,\phi)dt \\
&\quad +(\alpha+\beta)\int_0^T det\,R(t)\,(\sum_{k,l=1}^{3}\beta_{kl}(t)\frac{\partial z_k}{\partial y_l},\mathrm{div}\,\phi)dt = \int_0^T\ detR(t)(g_1,\phi)dt \\
&\quad\quad\quad +\chi\int_0^T det\,R(t)\,(\sum_{i=1}^{3}\nabla v_i A_i(t),\phi)dt, \qquad (3.35)
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^T det\,R(t)\,(h,\psi_t)dt + \nu\sum_{j=1}^{3}\int_0^T\int_\Omega det\,R(t)\,\sum_{k,l=1}^{3}\beta_{kl}(t)\frac{\partial h_j}{\partial y_l}\frac{\partial \psi_j}{\partial x_k}dydt \\
&\quad +\sum_{k,j=1}^{3}\int_0^T\int_\Omega det\,R(t)\,(h_k\frac{\partial \psi_j}{\partial x_k}v_j)dydt \\
&\quad = \sum_{k,j=1}^{3}\int_0^T\int_\Omega det\,R(t)\,(v_k\frac{\partial \psi_j}{\partial x_k}h_j)dydt. \qquad (3.36)
\end{aligned}
$$

Let us now consider the transformation $\Phi^{-1} : U \to Q$ defined by

$$\Phi^{-1}(y,t) = (yR(t),t).$$

We observe that det(J Φ^{-1}) is det $R^{-1}(t)$. Consequently, from (3.1) and by change of variables in the integrals (3.34)-(3.36), become

$$
\begin{aligned}
&-\int_Q u\ \varphi_t\, dxdt + (\mu+\chi)\sum_{k,j=1}^{3}\int_Q \frac{\partial u_j}{\partial x_k}\frac{\partial \varphi_j}{\partial x_k}\, dxdt - \sum_{k,j=1}^{3}\int_Q u_k\frac{\partial \varphi_j}{\partial x_k}u_j\, dxdt \\
&\qquad = \int_Q f\ \varphi\, dxdt - r\sum_{k,j=1}^{3}\int_Q b_k\frac{\partial \varphi_j}{\partial x_k}b_j\, dxdt + \chi\int_Q \operatorname{rot} w\ \varphi\, dxdt, \\
&-j\int_Q w\ \phi_t\, dxdt + \gamma\sum_{k,j=1}^{3}\int_Q \frac{\partial w_j}{\partial x_k}\frac{\partial \phi_j}{\partial x_k}\, dxdt - j\sum_{k,j=1}^{3}\int_Q u_k\frac{\partial \phi_j}{\partial x_k}w_j\, dxdt \\
&\qquad +2\chi\int_Q w\ \phi\, dxdt + (\alpha+\beta)\int_Q \operatorname{div} w\ \operatorname{div}\phi\, dxdt \\
&\qquad = \int_Q g\ \phi\, dxdt + \chi\int_Q \operatorname{rot} u\ \phi\, dxdt, \\
&-\int_Q b\ \psi_t\, dxdt + \nu\sum_{k,j=1}^{3}\int_Q \frac{\partial b_j}{\partial x_k}\frac{\partial \psi_j}{\partial x_k}\, dxdt - \sum_{k,j=1}^{3}\int_Q u_k\frac{\partial \psi_j}{\partial x_k}b_j\, dxdt \\
&\qquad + \sum_{k,j=1}^{3}\int_Q b_k\frac{\partial \psi_j}{\partial x_k}u_j\, dxdt = 0.
\end{aligned}
$$

which proves that (u, w, b) is a weak solution of (1.1)-(1.3), since the mappings

$$
\begin{aligned}
L^2(0,T;V(\Omega)) &\longrightarrow L^2(0,T;V(\Omega_t)) \\
v(y,t) &\longrightarrow u(x,t) = v(xR^{-1}(t),t) \\
h(y,t) &\longrightarrow b(x,t) = h(xR^{-1}(t),t) \\
L^2(0,T;H_0^1(\Omega)) &\longrightarrow L^2(0,T;H_0^1(\Omega_t)) \\
z(y,t) &\longrightarrow w(x,t) = z(xR^{-1}(t),t) \\
L^\infty(0,T;H(\Omega)) &\longrightarrow L^\infty(0,T;H(\Omega_t)) \\
v(y,t) &\longrightarrow u(x,t) = v(xR^{-1}(t),t) \\
h(y,t) &\longrightarrow b(x,t) = h(xR^{-1}(t),t) \\
L^\infty(0,T;L^2(\Omega)) &\longrightarrow L^\infty(0,T;L^2(\Omega_t)) \\
z(y,t) &\longrightarrow w(x,t) = z(xR^{-1}(t),t)
\end{aligned}
$$

are smooth bijections of class C^1, it follows that

$$
\begin{aligned}
&u,\, b \in L^2(0,T;V(\Omega_t)) \cap L^\infty(0,T;H(\Omega_t)), \\
&w \in L^2(0,T;H_0^1(\Omega_t)) \cap L^\infty(0,T;L^2(\Omega_t)).
\end{aligned}
$$

Finally a standard arguments show that $u(0) = u_0, w(0) = w_0$ and $b(0) = b_0$. Assertions (2.2) and (2.3) are proved analogously as in the case of the classical Navier-Stokes equations, see for instance, Lions [10]. This finished the proof of theorem.

□

References

[1] Ahmadi, G. and Shanhinpoor, M., *Universal stability of magneto-micropolar fluid motions*, Int. J. Enging. Sci., 12 (1994), 657-663.

[2] Conca, C. and Rojas-Medar, M.A., *The initial value problem for the Boussinesq equations in a time-dependent domain*, technical report, Universidad de Chile, 1993.

[3] Condiff, D.W. and Dalher, J.S., *Fluid mechanics aspects of antisymetric stress*, Phys. Fluid V. 7, Nº 6 (1964), 842-854.

[4] Dal Passo, R. and Ughi, M., *Problème de Dirichlet pour une classe d`equations paraboliques non lineares dans des ouverts non cylindriques*, Cras-Paris, Série I, T-308 (1989), 555-558.

[5] Eringen, A.C., *Theory of micropolar fluids*, J. Math. Mech. 16 (1996), 1-8.

[6] Eringen, A.C., *Simple microfluids*, Int. J. Enging. Sci., 2 (1964), 205-217.

[7] Fujita, H. and Sauer, N., *Construction of weak solutions of the Navier-Stokes equation in a noncylindrical domain*, Transactions AMS 75(1969), 465-468.

[8] Galdi, G.P. and Rionero, S., *A note on the existence and uniqueness of solutions of the micropolar fluid equations*, Int. J. Enging. Sci., 15 (1977), 105-108.

[9] Lions, J.L., *Une remarque sur les problèmes d'evolution non linéaires dans des domains non cylindriques*, Rev. Roumaine Math. Pures Appl., 9 (1964), 11-18.

[10] Lions, J.L., *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod Gauthier-Villars, Paris, 1969.

[11] Lukaszewicz, G. *On nonstationary flows of asymmetric fluids*, Rend. Accad. Naz. Sci. XL, Mem. Math., 106, vol. XII (1988), 83-97.

[12] Lukaszewicz, G., *On the existence, uniqueness and asymptotic properties of solutions of flows of asymmetric fluids*, Rend. Accad. Naz. Sci. XL, Mem. Math., Nº 107. Vol. XIII (1989), 105-120.

[13] Milla-Miranda, M. and Medeiros, L.A., *Exact controllability for Schöndringer equations in non cylindrical domains*, 41º Seminário Brasileiro de Análise (Minicurso), 1995.

[14] Milla-Miranda, M. and Medeiros, L.A., *Contrôlabilité exacte de l'equation de Schrödinger dans des domaines non cylindriques*, C.R. Acad. Sci. Paris 319(1994), 685-689.

[15] Milla-Miranda, M. and Límaco, J., *The Navier-Stokes equation in non-cylindrical domains*, Atas do 41º Seminário Brasileiro de Análise, 1995.

[16] Morimoto, H., *On the existence of periodic weak solutions of the Navier-Stokes equations in regions with periodically moving boundaries*, J. Fac. Sci. Univ. Tokyo Sect. IA 18, 499-524.

[17] Ortega-Torres, E.E. and Rojas-Medar, M.A., *Magneto-micropolar fluid motion: Global existence of strong solutions*, Conference in the IV Congresso Franco-Latinoamericano de Matemáticas Aplicadas, Métodos Numéricos en Mecánica, Concepción, Chile (1995).

[18] Ortega-Torres, E.E. and Rojas-Medar, M.A., *On the Uniqueness and Regularity of the Weak Solution for Magneto-Micropolar Fluid Equations*, Revista de Matemáticas Aplicadas, Univ. de Chile, 17 (1996), 75-90.

[19] Rojas-Medar, M.A., *Magneto-micropolar fluid motion: Existence and uniqueness of strong solutions*, Mathematische Nachrichten, 188 (1997), 301-319.

[20] Rojas-Medar, M.A. and Beltran-Barrios R. *The initial value problem for the equations of magnetohydrodynamic type in non-cylindrical domains*, Revista de Matemática de la Universidad Complutense de Madrid, 8(1), (1995), 229-251.

[21] Rojas-Medar, M.A. and Boldrini J.L., *Magneto-micropolar fluid motion: Existence of weak solutions*, Relatorio de pesquisa N$^{\underline{o}}$ 42/95, IMECC-UNICAMP (1995). To appear in Rev. Mat. Univ. Complutense de Madrid.

[22] Salvi, R. *On the existence of weak solution of a non-linear mixed problem for the Navier-Stokes equations in a time dependent domain*, J. Fac. Sci. Univ. Tokyo, Sect IA, Math. 32 (1985), 213-221.

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