DARBOUX TRANSFORMATIONS AND DARBOUX COVERINGS: SOME APPLICATIONS TO THE KP HIERARCHY*

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Abstract

Using the geometrical concept of Darboux covering we survey in a unified manner results connecting the Kadomtsev-Petviashvili (KP) hierarchy of integrable PDEs and the Darboux method. After reviewing the Darboux-KP (DKP) hierarchy, which was recently introduced by some of the authors, we show that suitable reductions thereof lead to rational reductions of the KP hierarchy. We also show that the KP hierarchy is a projection of the DKP hierarchy, the modified KP hierarchy is a restriction of KP to a suitable invariant manifold and that a certain discrete version of the KP equations can be obtained as iterations of the DKP ones.

Resumo

Fazendo uso do conceito geométrico de recobrimento de Darboux, apresentamos de forma unificada resultados ligando a hierarquia de EDPs completamente integráveis de Kadomtsev-Petviashvili (KP) e o método de Darboux. Após revermos a hierarquia de Darboux-KP, que foi recentemente definida por alguns dos presentes autores, mostramos que reduções apropriadas da mesma levam a reduções racionais da hierarquia KP. Mostramos também que a hierarquia KP é uma projeção da DKP, que a hierarquia KP modificada é uma restrição da KP a uma variedade invariante apropriada, e que uma certa versão discreta das equações da KP pode ser obtida como iterações das equações da DKP.

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1. Introduction

Darboux transformations were introduced more than a century ago [9] in the context of differential geometry and ordinary differential equations. Then they were used (sometimes implicitly) in different fields, e.g., in the study [19] of the Huygens property of hyperbolic partial differential equations (for more informations see, e.g., [2,24]). In particular, Darboux transformations were rediscovered in connection with the theory of *soliton equations*, as a powerful method to construct exact solutions (see [23] and references cited therein). Soliton equations are a class of nonlinear partial differential equations admitting solutions in the shape of a solitary wave (with additional interaction properties, see [6]). They can be integrated by means of the inverse scattering method, first described in [14]. Examples of soliton equations are

$$u_t - u_{xxx} + 6uu_x = 0 \qquad \text{(Korteweg–de Vries)}$$

$$u_t - u_{xxx} + 6u^2u_x = 0 \qquad \text{(modified KdV)}$$

$$u_{xt} + \sin u = 0 \qquad \text{(sine–Gordon)}$$

$$\mathrm{i}u_t + u_{xx} + 2|u|^2u = 0 \qquad \text{(nonlinear Schrödinger)}$$

$$u_{tt} - u_{xx} - u_{xxxx} + 3(u^2)_{xx} = 0 \qquad \text{(Boussinesq)}$$

$$(u_t + u_{xxx} - 6uu_x)_x + u_{yy} = 0 \qquad \text{(Kadomtsev–Petviashvili)}$$

In this paper we will present a geometrical description of the Darboux technique, allowing us to clarify some links among important elements of soliton theory [11,13,15], such as the Kadomtsev-Petviashvili (KP) hierarchy, the modified KP (mKP) hierarchy, the Miura map, and the discrete KP hierarchy (dKP), also called the generalized Toda lattice (see [17]). The results presented are part of a project aiming to give a coherent and geometric description of the integrability properties of soliton equations (for more informations see [8,20] and references therein; in particular, for a bihamiltonian approach to Darboux transformations see [22]). In particular, this paper may be considered as an introduction and a summary to the papers [7,21], were full proofs and details are given. However, there is some novelty, that will be pointed out through the paper. We would like to point out that, although familiarity with solitons is surely helpful as a motivation to the different equations under consideration, knowledge of soliton theory is not crucial to read this paper, which we tried to write as self–contained as possible.

The main points are the following: We introduce a geometrical setting for studying Darboux transformations, namely the concept of *Darboux covering*, and a new hierarchy, the Darboux-KP (DKP) hierarchy, which is a Darboux covering of the KP hierarchy. Then we show that suitable reductions of DKP lead to interesting reductions of KP (known in the literature [12,16] as *rational reductions* of KP). Finally, we show that DKP collects KP, mKP and dKP; more precisely, the KP hierarchy is a projection of DKP, the mKP hierarchy is a DKP restriction to a suitable invariant submanifold, and the discrete KP equations are obtained as iterations of the DKP ones.

2. Darboux coverings: a geometrical setting for Darboux transformations

Let $\mathcal{M}$ and $\mathcal{N}$ be two differentiable manifolds, X a vector field on $\mathcal{M}$, and Y a vector field on $\mathcal{N}$. We say that Y is a Darboux covering of X if there exist two (different) maps σ and μ from $\mathcal{N}$ to $\mathcal{M}$ mapping Y into X, that is, $\mu_*(Y) = \sigma_*(Y) = X$. In other words, integral curves of Y are sent into integral curves of X by means of two distinct maps.

Example 1. Let $\mathcal{F}$ be some space of C^∞–functions of one variable x (e.g., the space of C^∞–functions from the circle S^1 into $\mathbb{R}$), let $\mathcal{M} = \mathcal{F}$ and let X be the vector field on $\mathcal{M}$ associated to the Korteweg-de Vries (KdV) equation, that is,

$$X(u) = u_{xxx} - 6uu_x. \tag{1}$$

A Darboux covering of X can be constructed by putting $\mathcal{N} = \mathcal{F}$ and by taking

Y to be (the vector field associated to) the modified KdV (mKdV) equation,

$$Y(v) = v_{xxx} - 6v^2 v_x, \tag{2}$$

and

$$\begin{aligned} \mu(v) &= v^2 + v_x \\ \sigma(v) &= v^2 - v_x. \end{aligned} \tag{3}$$

This is easily seen by checking that, if v is a solution of mKdV, then $\mu(v)$ and $\sigma(v)$ are both solutions of KdV.

A Darboux covering of a vector field X can be used to generate new integral curves of X from old ones (i.e., new solutions of the corresponding differential equation from known solutions). To this aim, let us suppose $\mathcal{N}$ to be a vector bundle over $\mathcal{M}$, and μ the canonical projection of the bundle. Let $(x_1, \ldots, x_m, a_{m+1}, \ldots, a_n)$ be local fibered coordinates on $\mathcal{N}$; this means that $(x_1, \ldots, x_m)$ are coordinates on $\mathcal{M}$, and that the local expression of μ is

$$\mu : (x_1, \ldots, x_m, a_{m+1}, \ldots, a_n) \mapsto (x_1, \ldots, x_m).$$

Then the condition $\mu_*(Y) = X$ entails that the vector field Y has the form

$$\dot{x}^j = X^j(x) \tag{4}$$

$$\dot{a}^k = Y^k(x, a) \; , \tag{5}$$

where the X^j are the components of the vector field X. If a solution $x(t)$ of (4) is known, then it can be lifted into a solution $(x(t), a(t))$ of Y by solving the auxiliary system (5), controlled by $x(t)$. Once these equations have been solved, the map $\sigma : \mathcal{N} \to \mathcal{M}$ produces a second integral curve

$$\widetilde{x}(t) = \sigma(x(t), a(t)) \tag{6}$$

of the vector field X, depending on as many arbitrary parameters as there are arbitrary constants entering into the solution of the auxiliary system (5).

Another use of Darboux coverings which will be considered in this paper is to construct invariant submanifolds of a vector field. Using the notation introduced above, let $\mathcal{S} \subset \mathcal{N}$ be invariant for Y (that is, Y is tangent to $\mathcal{S}$). Then, both $\mu(\mathcal{S})$ and $\sigma(\mathcal{S})$ are invariant for X. In particular, if $\sigma(\mathcal{S}) \subset \mu(\mathcal{S})$, we have a Darboux covering for the restriction of X to $\mu(\mathcal{S})$. In Section 4 we will see that this (quite trivial) remark has nontrivial consequences, namely the existence of the so–called rational reductions of the KP hierarchy.

3. The DKP hierarchy: a Darboux covering for KP

In this section we will introduce a Darboux covering for the KP hierarchy, which is one of the most important hierarchies in the theory of soliton equations [10,11]. We will deal with the KP equations in a form which is natural from the point of view of the bihamiltonian approach to soliton equations. We refer to [8,20] for more informations and for the proofs of some assertions of this section. Let again $\mathcal{F}$ be some space of C^∞–functions of one variable x, and h a Laurent series of the form

$$h(z) = z + \sum_{j\geq 1} h_j z^{-j}, \tag{7}$$

whose coefficients h_j belong to $\mathcal{F}$. Then we define the Faà di Bruno iterates of h as

$$\begin{aligned} h^{(0)} &= 1 \\ h^{(j+1)} &= (\partial_x + h)h^{(j)} \ , \ \forall j \geq 0 \ . \end{aligned} \tag{8}$$

Notice that this recursion relations can also be solved backwards, in order to define (algebraically) $h^{(-1)}, h^{(-2)}, \ldots$ and to obtain a basis $\{h^{(j)}\}_{j\in\mathbb{Z}}$, with $h^{(j)} = z^j + O(z^{j-1})$, in the space of Laurent series. Let H_+ (resp. H_-) be the linear space spanned (over $\mathcal{F}$) by $\{h^{(j)}\}_{j\geq 0}$ (resp. by $\{h^{(j)}\}_{j<0}$). Notice that H_- is also spanned by $\{z^j\}_{j<0}$. Finally, define the canonical projections π_+ and π_- associated with H_+ and H_-. At this point we can introduce the *KP currents* as

$$H^{(j)} = \pi_+(z^j). \tag{9}$$

This means that $H^{(j)}$ is the unique linear combination of the Faà di Bruno iterates of h,

$$H^{(j)} = h^{(j)} + \sum_{l=0}^{j-2} p_l^j[h]h^{(l)}, \tag{10}$$

with the asymptotic behavior $H^{(j)} = z^j + O(z^{-1})$ as $z \to \infty$.

Example 2. It is clear that $H^{(1)} = h$. Let us compute the second current $H^{(2)}$,

$$H^{(2)} = h^{(2)} + c_1 h^{(1)} + c_0 h^{(0)}. \tag{11}$$

Since

$$\begin{aligned} h^{(1)} &= h = z + h_1 z^{-1} + h_2 z^{-2} + \dots \\ h^{(2)} &= h_x + h^2 = z^2 + 2h_1 + (h_{1x} + 2h_2)z^{-1} + (h_{2x} + h_1^2 + 2h_3)z^{-2} + \dots \end{aligned}$$

we obtain $H^{(2)} = z^2 + c_1 z + (2h_1 + c_0) + O(z^{-1})$. Therefore the asymptotic condition $H^{(2)} = z^2 + O(z^{-1})$ entails

$$c_1 = 0, \qquad c_0 = -2h_1, \tag{12}$$

so that

$$H^{(2)} = h^{(2)} - 2h_1 h^{(0)} = z^2 + (h_{1x} + 2h_2)z^{-1} + (h_{2x} + h_1^2 + 2h_3)z^{-2} + \dots . \tag{13}$$

In the same way one computes

$$\begin{aligned} H^{(3)} &= h^{(3)} - 3h_1 h^{(1)} - 3(h_2 + h_{1x})h^{(0)} \\ H^{(4)} &= h^{(4)} - 4h_1 h^{(2)} - (6h_{1x} + 4h_2)h^{(1)} - (4h_3 + 4h_{1xx} + 6h_{2x} - 2h_1^2)h^{(0)}. \end{aligned}$$

Having defined the currents $H^{(j)}$, we can now write the *KP equations* as the conservation laws

$$\partial_{t_j} h = \partial_x H^{(j)}, \tag{14}$$

to be thought of a family of vector fields on the space S_z^0 of Laurent series of the form (7). It can be shown that these vector fields commute.

For the reader who is familiar with soliton equations, we are going to show how to compare this definition of the KP hierarchy with the usual one. Let us develop z on the Faà di Bruno basis,

$$z = h^{(1)} + \sum_{j\geq 1} u_j h^{(-j)}, \tag{15}$$

in such a way to define a new set of variables $\{u_j\}_{j\geq 1}$ which are related to the coefficients $\{h_j\}_{j\geq 1}$ by an invertible transformation. If one introduces the pseudodifferential operator

$$\mathcal{L} = \partial_x + \sum_{j\geq 1} u_j \partial_x^{-j}, \tag{16}$$

then one can show that the KP equations (14) take the Lax form

$$\frac{\partial \mathcal{L}}{\partial t_j} = [\mathcal{L}, (\mathcal{L}^j)_+], \tag{17}$$

where $(\mathcal{L}^j)_+$ denotes the purely differential part of the pseudodifferential operator $\mathcal{L}^j$.

A Darboux covering for the KP hierarchy is provided by the Darboux-KP (DKP for short) hierarchy. In this case $\mathcal{M} = S_z^0$ and $\mathcal{N} = S_z^0 \times S_z$, where S_z is the space of monic Laurent series of degree 1, with coefficients in $\mathcal{F}$. Elements in $\mathcal{N}$ will be denoted with (h, a), where

$$a = z + \sum_{k\geq 0} a_k z^{-k}, \qquad a_k \in \mathcal{F}. \tag{18}$$

The map μ is simply the projection onto the first factor, $\mu(h, a) = h$, while $\sigma(h, a) = h + \partial_x \log a$. We will call μ and σ the Miura and the Darboux map of the KP theory. Finally, the DKP equations are the equations

$$\begin{cases} \partial_{t_j} h &= \partial_x H^{(j)} \\ \partial_{t_j} a &= a(\widetilde{H}^{(j)} - H^{(j)}) \end{cases} \tag{19}$$

where $\widetilde{H}^{(j)}$ is the current $H^{(j)}$ evaluated at the point $\tilde{h} = \sigma(h, a)$. In order to show that the DKP hierarchy gives a Darboux covering of the KP hierarchy, one simply has to check that solutions of DKP are mapped into solutions of KP by μ (trivial) and σ (easy).

Remark 3. It is worthwhile to notice that the DKP equations can be written in such a way to avoid the computation of $\widetilde{h}$ and of the corresponding currents. This can be done as follows. First of all, we remark that the relation $\widetilde{h} = h + \partial_x \log a$ is equivalent to the operator relation $a(\partial_x + \widetilde{h}) = (\partial_x + h)a$. This implies that $a(\partial_x + \widetilde{h})^k = (\partial_x + h)^k a$, so that $a\widetilde{H}^{(j)}$ is in the linear span of $\{(\partial_x + h)^k a\}_{k\geq 0}$. Hence $a\widetilde{H}^{(j)}$ can be characterized as the unique element $D^{(j)}$ in this span having the asymptotic behavior $D^{(j)} = z^{j+1} + a_0 z^j + \ldots + a_{j-1}z + O(z^0)$, and the second DKP equation can be written as

$$\partial_{t_j} a + aH^{(j)} = D^{(j)}, \tag{20}$$

without using the map σ.

4. Invariant submanifolds of DKP and rational reductions of KP

In this section we will give an example of the use of Darboux coverings to construct invariant submanifolds for a given vector field [7]. More precisely, we will construct the rational reductions of the KP hierarchy. These reductions give rise to the so–called constrained KP hierarchies recently studied both in the mathematical (see, e.g., [12,16]) and in the physical literature (see [1,4,5] and references quoted therein). From the point of view of Darboux coverings, the rational reductions of KP are simply the projections of very natural constraints on the DKP equations. To show this, let us introduce, for $l = 0, 1, \ldots$, the subset $\mathcal{S}_l$ of $\mathcal{N}$ defined as

$$\mathcal{S}_l = \{(h, a) \in \mathcal{N} \mid \pi_-(z^l a) = 0\} = \{(h, a) \in \mathcal{N} \mid z^l a \in H_+\}, \tag{21}$$

where π_- and H_+ have been defined in the previous section. In [21] the following result was proved.

Theorem 4. *For each $l = 0, 1, \ldots$ the submanifold $\mathcal{S}_l$ is invariant for the DKP equations. In other words, the DKP vector fields are tangent to $\mathcal{S}_l$.*

In order to write explicitly the equations of $\mathcal{S}_l$, let us note that the currents $H^{(j)}$, $j \geq 0$, form a basis of H_+. Keeping in mind their asymptotic behavior, it is easily shown that $z^l a \in H_+$ means that

$$z^l a = H^{(l+1)} + \sum_{m=0}^{l} a_m H^{(l-m)}, \tag{22}$$

or

$$a_{l+k} = H^{l+1,k} + \sum_{m=0}^{l} a_m H^{l-m,k}. \tag{23}$$

This shows that $\mathcal{S}_l$ may be parametrized by the components of the Laurent series $h(z)$ and by the first $(l+1)$ components $(a_0, a_1, \ldots, a_l)$ of $a(z)$. Thus $\mu(\mathcal{S}_l) = S^0_z$, the whole phase space of KP, and therefore no reduction of KP is obtained this way. Nevertheless, $\mathcal{S}_l \cap \mathcal{S}_{l+n}$ is also invariant for the DKP hierarchy, and $S'_{l,l+n} := \mu(\mathcal{S}_l \cap \mathcal{S}_{l+n})$ is a proper subset of S^0_z. The explicit equations of $S'_{l,l+n}$ can be obtained by writing the pair of equations

$$\begin{aligned} z^l a &= H^{(l+1)} + \sum_{k=0}^{l} a_k H^{(l-k)} \\ z^{l+n} a &= H^{(l+n+1)} + \sum_{k=0}^{l+n} a_k H^{(l+n-k)}, \end{aligned}$$

and by eliminating a, to get

$$H^{(l+n+1)} + \sum_{k=0}^{l+n} a_k H^{(l+n-k)} = z^n \big(H^{(l+1)} + \sum_{k=0}^{l} a_k H^{(l-k)}\big). \tag{24}$$

Since the map $\mu : \mathcal{N} \to \mathcal{M}$ is simply given by the canonical projection $\mu(h, a) = h$, equation (24) may be seen as the equation defining $S'_{l,l+n}$ in S^0_z. It can be checked that it gives h_j, for $j \geq l+n+1$, in terms of $a_0, \ldots, a_l$ and $h_1, \ldots, h_{l+n}$.

Example 5. Let us consider the simple intersection

$$S_{0,1} = \mathcal{S}_0 \cap \mathcal{S}_1. \tag{25}$$

As we shall see, it leads to the AKNS system, which is known to encompass both the nonlinear Schrödinger and the mKdV equations. The constraints of $S_{0,1}$ are

$$\left\{\begin{array}{rcl} a &=& H^{(1)}+a_0 \\ \\ za &=& H^{(2)}+a_0H^{(1)}+a_1\ , \end{array}\right. \tag{26}$$

and in this case (24) takes the form

$$z(h+a_0)=h_x+h^2+a_0h+a_1, \tag{27}$$

where we have used the computations done in Example 2. This equation can be solved in terms of (a_0,h_1) or also (a_0,a_1). If we choose this second possibility, we get

$$\begin{array}{rcl} h_1 &=& a_1 \\ h_2 &=& -(a_{1,x}+a_0a_1) \\ h_3 &=& a_{1,xx}+a_{0,x}a_1+2a_0a_{1,x}-a_1^2+a_0^2a_1, \end{array}$$

and so on. Substituting these constraints into the first two DKP equations we obtain

$$\begin{array}{l} \dfrac{\partial a_0}{\partial t_2}=(2a_1+a_{0,x}-a_0^2)_x \\ \\ \dfrac{\partial a_1}{\partial t_2}=-(a_{1,x}+2a_0a_1)_x \end{array} \tag{28}$$

and

$$\begin{array}{l} \dfrac{\partial a_0}{\partial t_3}=(a_{0,xx}-3a_0a_{0,x}+a_0^3-6a_0a_1)_x \\ \\ \dfrac{\partial a_1}{\partial t_3}=(a_{1,xx}+3a_0a_{1,x}-3a_1^2+3a_0^2a_1)_x. \end{array} \tag{29}$$

They coincide with the t_2 and t_3 flows of the so–called $(1|1)$–KdV theory [3,4], which gives, after the identifications

$$a_0=-\frac{r_x}{r},\quad a_1=-rq, \tag{30}$$

the classical AKNS hierarchy.

In terms of the coordinates u_i (i.e., of the pseudodifferential operator $\mathcal{L}$) introduced in Section 3, the equations of the submanifold $S'_{l,l+n}$ can be written (see [7] for the proof) in the form

$$\mathcal{L}^n = (L_{(l+1)})^{-1} L_{(l+n+1)}, \tag{31}$$

where $L_{(l+1)}$ and $L_{(l+n+1)}$ are differential operator of order $(l+1)$ and $(l+n+1)$. This is the constraint on $\mathcal{L}$ considered for the rational reductions of the KP hierarchy. Hence the restrictions to the submanifolds $S'_{l,l+n}$ give exactly the rational reductions of KP.

5. Elementary Darboux transformations and the modified KP hierarchy

In this section we describe in detail the simplest restriction of the DKP hierarchy, i.e., the case of

$$\mathcal{S}_0 = \{(h,a) \in \mathcal{N} \mid a \in H_+\} = \{(h,a) \in \mathcal{N} \mid a = h + a_0\}. \tag{32}$$

The elements of $\mathcal{S}_0$ can be parametrized by $a(z)$, so that $h = a - a_0$. Hence the submanifold $\mathcal{S}_0$ is diffeomorphic to S_z, and the restriction of the DKP equations to $\mathcal{S}_0$ gives evolution equations on a. We will write explicitly these equations, and show that they coincide with the modified KP (mKP) equations, using the second form of the DKP equations derived at the end of the previous section. For a proof which relies on the first form of DKP see [21].

All we have to do is to write the equation

$$\partial_{t_j} a + aH^{(j)} = D^{(j)} \tag{33}$$

solely in terms of a. Remember that $H^{(j)}$ is the KP current of $h = a - a_0$ and $D^{(j)}$ is the unique element in the span of $\{(\partial_x + h)^k a\}_{k\geq 0}$ having the asymptotic behavior $D^{(j)} = z^{j+1} + a_0 z^j + \ldots + a_{j-1} z + O(z^0)$. The key point is to introduce the Faà di Bruno iterates $a^{(j)}$ of a defined by

$$\begin{aligned} a^{(0)} &= 1 \\ a^{(j+1)} &= (\partial_x + a) a^{(j)} \ , \ \forall j \geq 0 \ , \end{aligned} \tag{34}$$

and to combine them in such a way to define the currents $A^{(j)}$ associated with a. They are the unique linear combinations of the iterates $\{a^{(j)}\}_{j\geq 1}$ of a with the asymptotic behavior

$$A^{(j)} = z^j + O(z^0)$$

as $z \to \infty$. Then, we can show

Proposition 6. *With the above notation, we have that*

$$\begin{aligned} H^{(j)} &= A^{(j)} - A^{j,0} & (35) \\ D^{(j)} &= A^{(j+1)} + \sum_{l=0}^{j-1} a_l A^{(j-l)}, & (36) \end{aligned}$$

where $A^{j,0}$ is the 0-th order coefficient of $A^{(j)}$ (i.e., $A^{(j)} = z^j + \sum_{l\geq 0} A^{j,l} z^{-l}$).

Proof: Since $a = h + a_0$, it is not difficult to check that the iterates $a^{(k)}$ of a belong to $H_+ = \operatorname{span}\{h^{(l)} \mid l \geq 0\}$. Therefore $A^{(j)}$ belong to H_+, and so $A^{(j)} - A^{j,0}$; but the asymptotic behavior of the latter is $z^j + O(z^{-1})$, hence it must coincide with the KP current $H^{(j)}$, showing (35).
As far as (36) is concerned, one immediately realizes that both members have the same asymptotic behavior, namely $z^{j+1} + a_0 z^j + \ldots + a_{j-1} z + O(z^0)$. Moreover, since for all $k \geq 1$ the iterates $a^{(k)}$ belong to the span of $\{(\partial_x + h)^l a \mid l \geq 0\}$, we also have that the mKP currents $A^{(k)}$ belong to this subspace. Then (36) follows from the characterization of $D^{(j)}$.

□

Theorem 7. *The restriction to $\mathcal{S}_0$ of the DKP hierarchy takes the form*

$$\partial_{t_j} a = \partial_x A^{(j)} \ , \tag{37}$$

where the $A^{(j)}$ are the mKP currents previously defined.

Proof: Since the operator $\partial_x + a$ sends the span of $\{a^{(l)}\}_{l\geq 1}$ into itself, and the mKP currents form a basis in such a span, we have that

$$(\partial_x + a) A^{(j)} = A^{(j+1)} + \sum_{l=0}^{j-1} a_l A^{(j-l)} + A^{j,0} a \ . \tag{38}$$

By eliminating $A^{(j+1)}$ between this equation and equation (36) we obtain

$$D^{(j)} = (\partial_x + a)A^{(j)} - A^{j,0}a. \tag{39}$$

Finally, the DKP equation (20) entails

$$\frac{\partial a}{\partial t_j} = -aH^{(j)} + D^{(j)} = -a(A^{(j)} - A^{j,0}) + (\partial_x + a)A^{(j)} - A^{j,0}a = \partial_x A^{(j)}, \tag{40}$$

and the proof is complete.

□

We can then conclude that the mKP hierarchy provides another Darboux covering of KP. The restrictions $\widehat{\mu}$ and $\widehat{\sigma}$ of the maps μ and σ to $\mathcal{S}_0$ are

$$h = \widehat{\mu}(a) = a - a_0 \tag{41}$$

and

$$\widetilde{h} = \widehat{\sigma}(a) = (a^{(2)} - a_0 a^{(1)})/a\ . \tag{42}$$

If $h = \widehat{\mu}(a)$ and $\widetilde{h} = \widehat{\sigma}(a)$ we will say that h and $\widetilde{h}$ are connected via the *elementary Darboux transformation* generated by a.

Another important fact is that we have a very simple splitting of mKP into KP plus one equation. Such property was already observed by Kupershmidt in [18]. However, we believe that in the present setting it takes a neater form. Indeed, from Theorem 7 it follows that $a = h + a_0$ is a solution of mKP if and only if h is a solution of KP and

$$\frac{\partial a_0}{\partial t_j} = \partial_x A^{j,0}. \tag{43}$$

Moreover, we can show in this simple example the use of Darboux coverings to construct new solutions (of KP, in this case). Suppose $h(t)$ to be a solution of KP; let $a_0(t)$ be a solution[1] of the "auxiliary problem" (43). Then $a(t) = h(t) + a_0(t)$ is a solution of mKP, and $\widetilde{h} = h + \partial_x \log a$ is a new solution of KP. Notice that $A^{j,0}$ is a differential polynomial in $(a_0, h_1, h_2, \ldots)$. Hence equation (43) is to be seen as an equation for a_0 which is "controlled" by the given solution $h(t)$ of KP.

[1]Such a solution can be, at least formally, constructed from $h(t)$ as $a_0(t) = -h(t, z_0)$, where z_0 is fixed. We will not prove this assertion.

6. Further reductions of mKP and the Gelfand–Dickey hierarchies

In the previous section we showed that the restriction of DKP to the invariant submanifold $\mathcal{S}_0$ is the mKP hierarchy, providing in this way another Darboux covering of KP. Now we will see that mKP admits a family of further restrictions, which are known as modified Gelfand–Dickey (mGD) hierarchies. This entails the existence of reductions of KP, the well–known Gelfand–Dickey (GD) hierarchies.

Consider the submanifold T_n of $\mathcal{S}_0$ defined by the constraint

$$A^{(n)} = z^n. \tag{44}$$

It can be shown [21] that this submanifold is invariant for the mKP hierarchy, i.e.,

$$\partial_{t_j}(A^{(n)} - z^n) = 0 \tag{45}$$

on T_n. The restriction to T_n of the mKP equations are the modified n–GD equations. We remark that on T_n the time t_n is stationary, i.e.,

$$\frac{\partial a}{\partial t_n} = 0 \tag{46}$$

as a consequence of the form (37) of the mKP equations.

Consider now the projection $T_n' = \mu(T_n)$ of T_n onto the phase space of the KP equations. Since h and a are related by $h = a - a_0$, from (35) we immediately have that

$$H^{(n)} = z^n \tag{47}$$

if h is in T_n'. The restriction to T_n' of the KP equations are the n–GD equations. Furthermore, since on T_n

$$0 = \partial_{t_n} a = a(\widetilde{H}^{(n)} - H^{(n)}), \tag{48}$$

we also see that

$$\widetilde{H}^{(n)} = z^n, \tag{49}$$

and, therefore, $\sigma(T_n)$ is contained in T'_n. We can then conclude that the mGD hierarchy is a Darboux covering of the GD hierarchy. The restriction to T_n of the maps

$$\begin{aligned} h &= \widehat{\mu}(a) = a - a_0 \\ \widetilde{h} &= \widehat{\sigma}(a) = h + \partial_x \log a \end{aligned}$$

are called the Miura map and the Darboux map of the GD equations.

Example 8 (KdV). The simplest case is $n = 2$,

$$A^{(2)} = z^2. \tag{50}$$

According to the definition of the mKP currents, this amounts to setting

$$a_x + a^2 - 2a_0 a = z^2 \tag{51}$$

on the Laurent series $a(z)$. This constraint allows to compute the coefficients a_j of $a(z)$, for $j \geq 1$, as differential polynomials of the first coefficient a_0. In particular we have

$$a_1 = \frac{1}{2}(-a_{0x} + a_0^2). \tag{52}$$

In the same way the constraint $H^{(2)} = z^2$ allows to compute all the coefficients h_j, for $j \geq 2$, as differential polynomials of the first coefficient h_1. The restriction to T_2 of the mKP hierarchy (that is, the modified 2–GD equations) is called the mKdV hierarchy; that of the KP hierarchy to T'_2 (that is, the 2–GD equations) is called the KdV hierarchy[2]. The restrictions of the maps $\widehat{\mu}$ and $\widehat{\sigma}$ to T_2 are given by

$$\begin{aligned} h_1 &= a_1 = \frac{1}{2}(-a_{0x} + a_0^2) \\ \widetilde{h}_1 &= h_1 + a_{0x}. \end{aligned}$$

In order to compare these expressions with the ones in Section 2, we introduce the variables $v = -a_0$ and $u = 2h_1$ used in the literature. Then the Miura map

[2]The KdV (resp. mKdV) equation written in the Introduction corresponds to the third time of the KdV (resp. mKdV) hierarchy.

and Darboux map take the usual form

$$u = v_x + v^2$$
$$\tilde{u} = u - 2v_x = -v_x + v^2$$

of the KdV theory.

We conclude this section with the remark that the GD reduction could be seen as a particular instance of the rational reductions discussed in Section 4. It is sufficient to consider also the submanifold

$$\mathcal{S}_{-1} = \{(h,a) \mid z^{-1}a \in H_+\} = \{(h,a) \mid a = z\}, \tag{53}$$

so that $\mathcal{S}_{-1} \cap \mathcal{S}_n = \{(h,a) \mid a = z \text{ and } z^n \in H_+\}$. But $z^n \in H_+$ implies that $H^{(n)} = z^n$, which is precisely the n–GD constraint.

7. Iterated Elementary Darboux Transformations and the discrete KP hierarchy

In Section 5 we have seen that, starting form a solution h of the KP hierarchy, one can generate a new solution $\widetilde{h}$ by means of an elementary Darboux transformation. Now we will iterate this procedure, and, as a last application of our construction, we will show how to make connection with the discrete KP hierarchy.

Suppose $a(0)$ to be a solution of mKP; then we know that $h(0) := \widehat{\mu}(a(0))$ and $h(1) := \widehat{\sigma}(a(0))$ are solutions of KP. Now, if we are able to solve the auxiliary problem (43) for $h = h(1)$, then we find another solution of mKP, say $a(1)$, such that $\widehat{\mu}(a(1)) = h(1)$. Hence we obtain a third solution $h(2) := \widehat{\sigma}(a(1))$ of KP. Suppose that we can iterate this process, in order to find a sequence $\{a(n)\}_{n\in\mathbb{Z}}$ of solutions of mKP fulfilling the *Darboux recursion relations*

$$\widehat{\mu}(a(n+1)) = \widehat{\sigma}(a(n)). \tag{54}$$

Then we will say that the $a(n)$ are the generators of a sequence of *iterated elementary Darboux transformations.* We are going to see that there is a very deep link between iterated elementary Darboux transformations and solutions of the discrete KP equations.

The procedure to construct the dKP equations is very similar to the one used to define the KP and the mKP hierarchies. We associate to any sequence $\{a(n)\}_{n\in\mathbb{Z}}$ of Laurent series of the form $a(n) = z + \sum_{l\geq 0} a_l(n) z^{-l}$ the discrete[3] Faà di Bruno iterates $a^{(j)}(n)$ defined by

$$\begin{aligned} a^{(0)}(n) &= 1 \\ a^{(j+1)}(n) &= a(n)a^{(j)}(n+1) \ . \end{aligned}$$

Then we combine these iterates in such a way to force the Laurent series

$$K^{(j)}(n) = a^{(j)}(n) + \sum_{l=0}^{j-1} r_l^j[a] a^{(l)}(n)$$

to have the asymptotic behavior $K^{(j)}(n) = z^j + O(z^{-1})$ as $z \to \infty$. The discrete KP equations are

$$\partial_{t_j} a(n) = a(n)(K^{(j)}(n+1) - K^{(j)}(n)) \ . \tag{55}$$

They can be shown to be equivalent to the equations given in [17] for a discrete Lax operator.

The key result to compare sequences of elementary Darboux transformations and solutions of dKP is the following one, for whose proof we refer to [21].

Theorem 9. *Let $\{a(n)\}_{n\in\mathbb{Z}}$ be a sequence of Laurent series of the form (18) satisfying the Darboux recursion relations (54), and let $h(n) = a(n) - a_0(n)$. Then, the differential KP currents $H^{(j)}(n)$ associated with $h(n)$ coincide with the discrete KP currents $K^{(j)}(n)$ attached to $a(n)$.*

Now, we can show that a sequence $\{a(n)\}_{n\in\mathbb{Z}}$ iterated elementary Darboux transformations is a solution of the discrete KP equations. Indeed, we recall

[3]We remark that in this case the coefficients $a_l(n)$ are *not* assumed to depend on a space variable x.

that the pair $(h(n), a(n))$ is a solution of DKP for each solution $a(n)$ of mKP. Therefore, the sequence $\{a(n)\}$ verifies the equations

$$\frac{\partial a(n)}{\partial t_j} = a(n)\left(H^{(j)}(n+1) - H^{(j)}(n)\right),$$

where $H^{(j)}(n)$ and $H^{(j)}(n+1)$ are the KP currents associated to the points $h(n)$ and $h(n+1)$, respectively. By Theorem 9, this sequence also verifies the discrete KP equations

$$\frac{\partial a(n)}{\partial t_j} = a(n)\left(K^{(j)}(n+1) - K^{(j)}(n)\right). \tag{56}$$

Conversely, let $\{a(n)\}$ be a solution of dKP, and let us set $x = t_1$ and $h(n) = a(n) - a_0(n)$. Then the first equation of the dKP hierarchy is

$$\frac{\partial a(n)}{\partial x} = a(n)(h(n+1) - h(n)), \tag{57}$$

that is, the Darboux recursion relation. At this point Theorem 9 can be used again to deduce that the discrete currents $K^{(j)}(n)$ coincide with the differential currents $H^{(j)}(n)$ associated with $h(n)$. Then $(h(n_0), a(n_0))$ solves for all n_0 the DKP equations

$$\frac{\partial a(n_0)}{\partial t_j} = a(n_0)\left(H^{(j)}(n_0+1) - H^{(j)}(n_0)\right) \tag{58}$$

as a consequence of (56). It follows that $a(n_0)$ is a solution of mKP, and that $\{a(n)\}$ is a sequence of elementary Darboux transformations. For a different proof of this fact, involving the evolution equations of the currents $A^{(j)}$, see [21].

References

[1] Aratyn, H., *Integrable Lax Hierarchies, their Symmetry Reductions and Multi-Matrix Models*, Lectures at VIII J.A. Swieca Summer School, Section: Particles and Fields, Rio de Janeiro, February 1995.

[2] Berest, Y.; Veselov, A., *Huygens' principle and integrability*, Russ. Math. Surveys 49, (1994), 5-77.

[3] Bonora, L.; Liu, Q.P.; Xiong, C.S., *The integrable hierarchy constructed from a pair of KdV-type hierarchies and its associated W–algebra*, Commun. Math. Phys. 175, (1996), 177-202.

[4] Bonora, L.; Xiong, C.S., *The (N-th,M-th) KdV hierarchy and the associated W–algebra*, J. Math. Phys. 35, (1994), 5781-5819.

[5] Bouwknegt, P.; Schoutens, K. (eds.), *W–symmetry*, Adv. Series in Math. Phys. vol. XX, World Scientific, Singapore, 1995.

[6] Calogero, F.; Degasperis, A., *Spectral Transform and Solitons*, North-Holland, Amsterdam, 1982.

[7] Casati, P.; Falqui, G.; Magri, F.; Pedroni, M., *Darboux Coverings and Rational Reductions of the KP Hierarchy*, Lett. Math. Phys. 41, (1997), 291-305.

[8] Casati, P.; Falqui, G.; Magri, F.; Pedroni, M., *Soliton equations, bihamiltonian manifolds, and integrability*, Notes of a course given at the 21st Brazilian Mathematical Colloquium (IMPA, Rio de Janeiro, July 1997), IMPA.

[9] Darboux, G., *Leçons sur la Théorie Générale de Surfaces et les Applications Géométriques du Calcul Infinitésimal, II Partie*, Gauthiers-Villars, Paris, 1889.

[10] Date, E.; Jimbo, M.; Kashiwara, M.; Miwa, T., *Transformation Groups for Soliton Equations*, Proceedings of R.I.M.S. Symposium on Nonlinear Integrable Systems (Jimbo, M., Miwa T., eds.), pp. 39-119, World Scientific, Singapore, 1983.

[11] Dickey, L.A., *Soliton Equations and Hamiltonian Systems*, Adv. Series in Math. Phys. vol. 12, World Scientific, Singapore, 1991.

[12] Dickey, L.A., *On the constrained KP Hierarchy*, Lett. Math. Phys. 35, (1995), 229-236.

[13] Faddeev, L.D.; Takhtajan, L.A., *Hamiltonian Methods in the Theory of Solitons*, Springer, 1987.

[14] Gardner, C.S.; Green, J.M.; Kruskal, M.D.; Miura, R.M., *Korteweg-de Vries Equation and Generalizations. VI. Methods for Exact Solution*, Comm. Pure Appl. Math. 27, (1974), 97-133.

[15] Konopelchenko, B.G., *Nonlinear integrable equations: recursion operators, group theoretical and Hamiltonian structures of soliton equations*, Springer-Verlag, Berlin, 1987.

[16] Krichever, I.M., *General rational reductions of the KP hierarchy and their symmetries*, Funct. Anal. Appl. 29, (1995), 75-80.

[17] Kupershmidt, B.A., *Discrete Lax Equations and Differential–Difference Calculus*, Astérisque, Paris, 1985.

[18] Kupershmidt, B.A., *Canonical property of the Miura maps between the mKP and KP hierarchies, continuous and discrete*, Commun. Math. Phys. 167, (1995), 351-371.

[19] Lagnese, J.E.; Stellmacher, K.L., *A method of generating classes of Huygens' operators*, J. Math. Mech. 17, (1967), 461-472.

[20] Magri, F., *Eight lectures on integrable systems*, to appear in the Proceedings of the CIMPA Summer School on Nonlinear Systems (B. Grammaticos and K. Tamizhmani, eds.), Pondicherry, India, January 1996.

[21] Magri, F.; Pedroni, M.; Zubelli, J.P., *On the geometry of Darboux transformations for the KP Hierarchy and its connection with the discrete KP hierarchy*, Commun. Math. Phys. 188, (1997), 305-325.

[22] Magri, F.; Zubelli, J.P., *Bihamiltonian formalism and the Darboux-Crum method. Part I: From the KP to the mKP hierarchy*, Inverse Problems 13, (1997), 755-780.

[23] Matveev, V.B.; Salle, M.A., *Darboux transformations and solitons*, Springer-Verlag, New York, 1991.

[24] Zubelli, J.P., *Topics on wave propagation and Huygens' principle*, Notes of a course given at the 21^{st} Brazilian Mathematical Colloquium (IMPA, Rio de Janeiro, July 1997), IMPA.

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