WHEN IS A POLYCYCLIC-BY-FINITE GROUP HYPERCENTRAL-BY-FINITE?

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Abstract

The set of the nilpotent-by-cyclic subgroups is a testfamily to decide whether a polycyclic-by-finite group is hypercentral-by-finite.

Resumo

O conjunto dos subgrupos nilpotentes-por-cíclicos forma uma família de teste para decidir se um grupo policíclico-por-finito é hipercentral-por-finito.

Let G be a group, $\mathcal{S}(G) = \{S/M \mid M \trianglelefteq S \leq G\}$ the set of its sections. Of some interest in group theory are theorems which guarantee a group theoretical property $\mathfrak{X}$ for G if $\mathfrak{X}$ is known for a testfamily $\mathcal{T}(G) \subseteq \mathcal{S}(\mathrm{G})$ which consists of some "small" members of $\mathcal{S}(G)$ - in particular when $\mathcal{T}(G)$ is a set of special subgroups or quotients of G. The purpose of this short communication is to add one more theorem of this type. If G is a group, let $\boldsymbol{Z}_\infty(G)$ denote its hypercenter. G is *hypercentral-by-finite* if $G/\boldsymbol{Z}_\infty(G)$ is finite. We want to prove:

Theorem. *Let G be a polycyclic-by-finite group. The following statements are equivalent:*

a) *G is hypercentral-by-finite.*

b) *H is hypercentral-by-finite for every nilpotent-by-cyclic subgroup H of G.*

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We use common notions and notation. For explanation see [2] and [3]. For the proof of our theorem we need the following generalization of a classical result due to Hirsch which tests the nilpotency of a polycyclic-by-finite group through its finite quotients (see [1], [3, p. 18, Thm. 2]):

Proposition. *Let G be a polycyclic-by-finite group and E a normal subgroup of G. The following statements are equivalent:*

a) $E \leq \boldsymbol{Z}_\infty(G)$.

b) $E/N \leq \boldsymbol{Z}_\infty(G/N)$ *for all* $N \trianglelefteq G$ *such that* $N \leq E$ *and* E/N *is finite.*

For $E = G$, Hirsch's theorem is obtained.

Proof: a) $\Rightarrow$ b) is immediate.
To see b) $\Rightarrow$ a), we may choose a polycyclic $M \trianglelefteq G$, $M \leq E$ such that E/M is finite. We induct on the derived length d of M to prove the assertion. For $d = 0$, E is finite and there is nothing to prove. If $d \geq 1$ and $M = M^{(0)} > M' > ... > M^{(d-1)} > M^{(d)} = 1$ is the derived series of M, we put $A = M^{(d-1)}$. Thus A is abelian and $A \trianglelefteq G$. Since the hypothesis in b) is inherited by G/A, we conclude $E/A \leq \boldsymbol{Z}_\infty(G/A)$ by induction. This means that the iterated commutator group $[E,\ _{n_1}G] \leq A$ for some n_1. Here $[E,\ _{n_1}G] = [...[[E,G],G],...,G]$, where G is written n_1 times. There is a torsion-free $S \trianglelefteq G$, $S \leq A$ and A/S finite. Since E/S is residually finite, there is $R \trianglelefteq E$, E/R finite, such that $S = A \cap R$. For $N = \bigcap_{g \in G} R^g$ we have $N \trianglelefteq G$, E/N finite and $S = A \cap N$. Since $E/N \leq \boldsymbol{Z}_\infty(G/N)$ by hypothesis, $[E,\ _{n_2}G] \leq N$ for some n_2. Therefore $[E,\ _nG] \leq S$ for $n = \max(n_1, n_2)$. Let S have rank r and let p be a prime. We have $|S/S^p| = p^r$ where S^p is the smallest subgroup of S with elementary abelian p-factor group. There is $X_p \trianglelefteq G$, $X_p \leq E$, E/X_p finite, such that $S^p = S \cap X_p$. Now $SX_p/X_p \leq E/X_p \leq \boldsymbol{Z}_\infty(G/X_p)$. Since SX_p/X_p and S/S^p are G-isomorphic, we have $[E,\ _{n+r}G] \leq S^p$. Since r does not depend upon p, we see that $[E,\ _{n+r}G] \leq \bigcap_p S^p = 1$. This means $E \leq \boldsymbol{Z}_\infty(G)$.

Lemma. *Let p be a prime and P a normal p-subgroup of the finite group G. $P \leq \boldsymbol{Z}_\infty(G)$, if and only if, P is centralized by all p'-elements of G.*

Proof: See [2, Kap. VI, Hilfssatz 12.9].

Proof of the Theorem:

a) $\Rightarrow$ b) is immediate.

b) $\Rightarrow$ a): Let M be a polycyclic normal subgroup of G such that G/M is finite. Let $F = \boldsymbol{F}(M)$ be the Fitting subgroup of M and $Y/F = \boldsymbol{F}(M/F)$. Let $Y = \langle y_1, ..., y_r\rangle$ and consider the subgroups $K_i = F\langle y_i\rangle$ for $i = 1, ..., r$. We have $K_i \trianglelefteq\trianglelefteq G$ and therefore $\boldsymbol{F}(K_i) = \boldsymbol{F}(M) = F$. The K_i are nilpotent-by-cyclic. Therefore $K_i/\boldsymbol{Z}_\infty(K_i)$ is finite by hypothesis. Since $\boldsymbol{Z}_\infty(K_i) \leq F$, also K_i/F is finite. So $Y/F = \langle K_1/F, ..., K_r/F\rangle$ is finite. Since $\boldsymbol{C}_{M/F}(Y/F) \leq Y/F$ (see [3, pg. 27, Ex. 2]), we conclude that G/Y is finite. Thus G/F is finite.

Let $G = Fx_1 \cup ... \cup Fx_n$ with $x_1, ..., x_n \in G$ and $n = |G/F|$. Thus $H_k = F\langle x_k\rangle$ are nilpotent-by-cyclic subgroups of G $(1 \leq k \leq n)$. Since $H_k/\boldsymbol{Z}_\infty(H_k)$ is finite by hypothesis, $|G : \boldsymbol{Z}_\infty(H_k)|$ is finite too. We put $D = \bigcap_{k=1}^n \boldsymbol{Z}_\infty(H_k)$ and have that $D_G = \bigcap_{g\in G} D^g$ is a normal subgroup of G of finite index. It suffices to show that D_G is contained in the hypercenter of G: Certainly D_G is nilpotent. Let $N \trianglelefteq G$ such that $N \leq D_G$ and D_G/N is finite. Let p be a prime number and P/N the Sylow-p-subgroup of D_G/N. We have $P/N \trianglelefteq G/N$. Let $y \in G$ such that yN is a p'-element of G/N and suppose $y \in H_k$, say. We apply now the above Lemma to G/N in both directions: Since $P/N \leq \boldsymbol{Z}_\infty(H_k/N)$, the element yN centralizes P/N. This means $P/N \leq \boldsymbol{Z}_\infty(G/N)$. Since this holds for all p, we see that $D_G/N \leq \boldsymbol{Z}_\infty(G/N)$. By our Proposition, $D_G \leq \boldsymbol{Z}_\infty(G)$.

References

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