ESTIMATES OF THE FIRST EIGENVALUE OF MINIMAL HYPERSURFACES OF $\mathbb{S}^{n+1}$

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Abstract

We consider a solution f of a certain Dirichlet Problem on a domain in $\mathbb{S}^{n+1}$ whose boundary is a minimal hypersurface and we prove a Poincaré-type inequality for f. Moreover we have an estimate for the first non zero eingenvalue for the closed eigenvalue problem on the boundary.

Sumário

Consideramos a solução f de um certo problema de Dirichlet em um domínio de $\mathbb{S}^{n+1}$ cuja fronteira é uma hipersuperfície mínima e provamos uma desigualdade do tipo Poincaré para f. Também mostramos uma estimativa para o primeiro auto-valor não nulo do Laplaciano (problema fechado) na fronteira.

1. Introduction

In this note we will let M^n be an embedded compact orientable minimal hypersurface in $\mathbb{S}^{n+1}$. Yau conjectured that the first nonzero eigenvalue $\lambda_1(M)$ for the closed eigenvalue problem $\Delta_M u + \lambda u = 0$ on M with the induced metric was equal to n. Observe that such M divides $\mathbb{S}^{n+1}$ into two connected components Ω_1 and Ω_2 such that $\partial\Omega_1 = \partial\Omega_2 = M$. Choi-Wang [CW] with a clever idea, applied Relly formula to the solution of the following Dirichilet problem,

$$\begin{cases} \overline{\Delta} f &= 0 \quad \text{on} \quad \Omega_1 \\ f &= \varphi \quad \text{on} \quad M\,, \end{cases} \tag{1}$$

where φ is the first eigenfunction for the closed eigenvalue problem on M to prove that $\lambda_1(M) > n/2$. We improve (conceptually) Choi-Wang's estimate in

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terms of the solution f of the problem (1) (see Corollary 1.2, inequality 3) with possibility to set up Yau's conjecture provided that one proves equality in (5). The symbols $\overline{\Delta}$ and $\overline{\nabla}$ will be respectively the laplacian and the gradient of the metric of $\mathbb{S}^{n+1}$ on Ω_1 while Δ and ∇ will be the laplacian and gradient of the induced metric on M.

Theorem 1.1. *Let f be the solution of the Dirichlet Problem (1) then*

$$\begin{aligned} p(t) &= (2\,\lambda_1(M) - n\,)\,\|\overline{\nabla}\,f\|_2^2\; t^2 + 2\,\lambda_1(M)\,\|f\|_2^2\; t \\ &\quad + \tfrac{n}{n+1}\,\|f\|_2^2 \;\geq\; 0 \quad \forall\, t\in\mathbb{R}, \end{aligned} \tag{2}$$

where $\|\,\|_2$ denotes the L^2 norm on Ω_1.

Corollary 1.2. *Let M^n be an orientable embedded minimal hypersurface of $\mathbb{S}^{n+1}$ and $\lambda_1(M)$ its first non-zero eigenvalue of the laplacian for the closed eigenvalue problem on M. Consider the problem (1) and f its solution. Then*

$$\lambda_1(M) \geq \frac{n}{2} + \frac{n}{2}\,\rho, \tag{3}$$

where

$$\begin{aligned} \rho &= \rho(\|f\|_2^2\,,\,\|\nabla f\|_2^2) \\ &= \left[\frac{2\,\|\overline{\nabla} f\|_2^2 - (n+1)\,\|f\|_2^2 - 2\,\|\overline{\nabla} f\|_2^2 \sqrt{1-(n+1)\,\dfrac{\|f\|_2^2}{\|\overline{\nabla} f\|_2^2}}}{(n+1)\,\|f\|_2^2} \right]. \end{aligned} \tag{4}$$

Observe that $0<\rho\leq 1$ and $\rho = 1 \Leftrightarrow \|\nabla f\|_2^2 = (n+1)\|f\|_2^2$.

Although the function f does not belong to $\mathrm{H}_0^1(\Omega_1)$ we have the following Poincaré type inequalities.

Corollary 1.3. *Let f be the solution of the Dirichlet Problem (1). Then f satisfies the following inequalities:*

$$\|\overline{\nabla}\,f\|_2^2 \;\geq\; (n+1)\,\|f\|_2^2, \tag{5}$$

$$\|\overline{D}^2 f\|_2^2 > \frac{n(n+1)}{4}\|f\|_2^2, \tag{6}$$

where $\overline{D}^2 f$ *is the Hessian of* f.

2. Proof of the Results

Let Ω be a Riemannian manifold of dimension n with smooth boundary $\partial\Omega$ and let f be a function on Ω which is smooth up to the boundary $\partial\Omega$. We denote $\varphi = f\mid_{\partial\Omega}$ and $u = \frac{\partial f}{\partial\nu}$, where $\frac{\partial f}{\partial\nu}$ is the normal outward derivative of f. For $X,\ Y \in T_p\Omega$, the Hessian tensor is denoted by $(\overline{D}^2 f)(X,Y)$, where X and Y are extended arbitrarily to a vector field near p . Let $B(v,w)$ be the second fundamental form of $\partial\Omega$ relative to Ω. Here v, w are tangent to $\partial\Omega$, H is the mean curvature of $\partial\Omega$, i.e. $n\,H = trB$ and Ric is the Ricci curvature of Ω. The following identity is known as the Reilly formula.

$$\begin{aligned} \textstyle\int_\Omega(\overline{\Delta} f)^2 &= \textstyle\int_\Omega|\overline{D}^2 f|^2 + \int_\Omega Ric(\overline{\nabla} f,\overline{\nabla} f) + \int_{\partial\Omega} 2u\Delta\varphi \\ &\quad \textstyle+ \int_{\partial\Omega} B(\nabla\varphi,\nabla\varphi) + \int_{\partial\Omega} nHu^2. \end{aligned} \tag{7}$$

Now we can show the proof of the main result. If $t = 0$ we are done. Now for $t \neq 0$ we consider the following Dirichlet Problem

$$\begin{cases} \overline{\Delta} g &= f, \text{ on } \Omega_1 \\ g &= t\,\varphi, \text{ on } M \end{cases} \tag{8}$$

Applying Green formula we obtain

$$\begin{cases} \int_M \varphi\,\frac{\partial f}{\partial\nu} &= \int_{\Omega_1}|\overline{\nabla} f|^2 \\ t\,\int_M \varphi\,\frac{\partial f}{\partial\nu} &= \int_{\Omega_1}\langle\overline{\nabla} f,\overline{\nabla} g\rangle \\ \int_M \varphi\,\frac{\partial g}{\partial\nu} &= \int_{\Omega_1} f^2 + \int_{\Omega_1}\langle\overline{\nabla} f,\overline{\nabla} g\rangle \end{cases} \tag{9}$$

From (9) we get

$$\int_{\Omega_1}\langle\overline{\nabla} f\,,\,\overline{\nabla} g\rangle = t\int_{\Omega_1}|\overline{\nabla} f|^2 \tag{10}$$

and by Cauchy-Schawarz inequality we get

$$\int_{\Omega_1} |\overline{\nabla} g|^2 \geq t^2 \int_{\Omega_1} |\overline{\nabla} f|^2 \tag{11}$$

Since $\int_{\Omega_1} \langle \overline{\nabla} f, \overline{\nabla} g \rangle = t \int_{\Omega_1} |\overline{\nabla} f|^2$ we have from the third equation in (9) that

$$t \int_M \varphi \frac{\partial g}{\partial \nu} = t \int_{\Omega_1} f^2 + t^2 \int_{\Omega_1} |\overline{\nabla} f|^2 \tag{12}$$

Now applying Reilly formula to g, using the fact that $|\overline{D}^2 g|^2 \geq \frac{1}{n+1}(\overline{\Delta} g)^2$ and the assumption that $\int_M B(\nabla \varphi \, \nabla \varphi) \geq 0$ we have

$$\frac{n}{n+1} \int_{\Omega_1} (\overline{\Delta} g)^2 \geq n \int_{\Omega_1} |\overline{\nabla} g|^2 + 2 \int_M \frac{\partial g}{\partial \nu} (\Delta t \varphi) \tag{13}$$

On the other hand taking in account that $\overline{\Delta} g = f$, $\int_{\Omega_1} |\overline{\nabla} g|^2 \geq t^2 \int_{\Omega_1} |\overline{\nabla} f|^2$ and

$$t \int_M \varphi \frac{\partial g}{\partial \nu} = t \int_{\Omega_1} f^2 + t^2 \int_{\Omega_1} |\overline{\nabla} f|^2$$

we have that (13) implies

$$\frac{n}{n+1} \int_{\Omega_1} f^2 \geq n t^2 \int_{\Omega_1} |\overline{\nabla} f|^2 - 2\lambda_1(M) \left[t \int_{\Omega_1} f^2 + t^2 \int_{\Omega_1} |\overline{\nabla} f|^2 \right] \tag{14}$$

Therefore we have

$$p(t) = (2\lambda_1(M) - n) \, \|\overline{\nabla} f\|_2^2 \, t^2 + 2\lambda_1(M) \, \|f\|_2^2 \, t + \frac{n}{n+1} \|f\|_2^2 \geq 0 \tag{15}$$

This finishes the proof of Theorem (1.1). Observe that $p(t) \geq 0$ therefore, its discriminant is non-positive. This can be read as follows

$$(2\lambda_1(M) - n) \geq \frac{n+1}{n} \lambda_1^2(M) \frac{\|f\|^2}{\|\overline{\nabla} f\|^2}. \tag{16}$$

Thus we have from (16) the inequality (3) and the following Poincaré inequality for f,

$$\|\overline{\nabla} f\|^2 \geq (n+1) \|f\|^2. \tag{17}$$

In the proof of Theorem (1.1) we did not count an extra term "$\int_M B(\nabla t\varphi, \nabla t\varphi) \geq 0$" on the right hand side of (13). If we did, we would have that $p(t) \geq \int B(\nabla \varphi, \nabla \varphi) \, t^2$. From that we would conclude that

$$(2\lambda_1(M) - n) \|\overline{\nabla} f\|_2^2 - \int B(\nabla \varphi, \nabla \varphi) \; \geq \frac{\lambda_1^2(M)(n+1)}{n} \|f\|_2^2.$$

On the other hand, Reilly Formula also gives $(2\lambda_1(M) - n)\|\overline{\nabla} f\|_2^2 = \|\overline{D}^2 f\|_2^2 + \int B(\nabla\varphi, \nabla\varphi)$. Therefore we obtain

$$\|\overline{D}^2 f\|_2^2 \geq \frac{\lambda_1^2(M)(n+1)}{n}\|f\|_2^2 > \frac{n(n+1)}{4}\|f\|_2^2, \tag{18}$$

since $\lambda_1(M) > \frac{n}{2}$.

References

[CW] Choi, H. I., Wang, A. N. : *A first eigenvalue estimate for minimal hypersurfaces.* J. Diff. Geom. 18, 559-562 (1983).

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