ON CONSTANT MEAN CURVATURE SPACELIKE HYPERSURFACES IN LORENTZ MANIFOLDS

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Abstract

In 1988 S. Montiel proved that the only constant mean curvature compact spacelike hypersurfaces in de Sitter space are the umbilical ones (conjectured by A. J. Goddard in 1977). Since then increased the quest for conditions in more general spacetimes in order to guarantee that constant mean curvature spacelike hypersurfaces are umbilical. This work presents a unified treatment of relevant facts and ideas on spacelike hypersurfaces of constant mean curvature in some Lorentz manifolds with a concise overview of results and methods in the last twenty years.

Resumo

Em 1988 S. Montiel provou que as únicas hipersuperfícies compactas tipo-espaço com curvatura média constante no espaço de de Sitter eram as umbílicas (conjecturado por A. J. Goddard em 1977). Desde então, cresceu o interesse em encontrar condições em espaços-tempo mais gerais para garantir que as hipersuperfícies tipo-espaço de curvatura média constante sejam umbílicas. Este artigo apresenta um tratamento unificado de fatos e idéias relevantes sobre hipersuperfícies tipo-espaço com curvatura média constante em algumas variedades de Lorentz, com uma visão sucinta de resultados e métodos nos últimos vinte anos.

1. Introduction

In 1977 A. J. Goddard [G] conjectured that the only complete constant mean curvature spacelike hypersurfaces (i.e, the induced metric is Riemannian) in the de Sitter space are the umbilical ones. The de Sitter space is the hyperquadric given by $S_1^{n+1} = \{u \in R_1^{n+2}; \langle u, u\rangle = 1\}$, where R_1^{n+2}, the Lorentz-Minkowski space, is the Euclidean space R^{n+2} with the metric given by $\langle v, w\rangle = -v_0w_0 +$

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$v_1w_1+...+v_{n+1}w_{n+1}$. The first result in the direction of Goddard's conjecture was obtained by J. Ramanathan [R] in 1987 : "if the mean curvature H of a complete surface in S_1^3 satisfies $H^2<1$, then the surface is umbilical"(Theorem 3.2 in Section 3). It was known since 1981 an example of Dajczer and Nomizu [D-N] of a constant mean curvature complete surface in S_1^3 which is not umbilical. Still in 1987 (and independently) K. Akutagawa [A] proved Goddard's conjecture when $H^2<1$, if $n=2$ and $H^2<4(n-1)/n^2$, if $n>2$ (Theorem 3.4 in Section 3). He also showed that when $n=2$, for any constant $H^2>c^2$ there exists a non-umbilical surface of mean curvature H in the de Sitter space $S_1^3(c)$ of constant curvature $c>0$. One year later S. Montiel [M1] solved Goddard's problem in the compact case in S_1^{n+1} without restriction over the range of H (Theorem 3.5 in Section 3). He also constructed examples of non-umbilical complete spacelike hypersurfaces in S_1^{n+1} with constant mean curvature H satisfying $H^2\geq 4(n-1)/n^2$, including the so called hyperbolic cylinders. On Goddard's conjecture, in a chronological sequence we have, in 1989, V. Oliker's work published in ([O], 1992). He obtained estimates of the spread of the principal curvatures on a compact spacelike hypersurface in S_1^{n+1}. Using such estimates he proved that if $H^2<4(n-1)/n^2$ and H is close to a constant then the hypersurface is close to being umbilical (see Theorem 3.10 in Section 3). Oliker also obtains, in particular, results in [A] and [M1] above mentioned (see Corollary 3.9 in Section 3). In [M2], for the bound of H in the above inequality, Montiel proved that the only complete spacelike hypersurface in S_1^{n+1} with constant mean curvature $H=2\sqrt{n-1}/n$ with more than one topological end is an hyperbolic cylinder. Recently, the authors [Br-C2] obtained a characterization of the hyperbolic cylinders with constant scalar curvature, under a sharp limitation of the mean curvature function (see Theorem 3.11 in Section 3).

We are leaving out the specific case of spacelike hypersurfaces with mean curvature $H=0$ in Lorentz manifolds. These are called maximal because in ambient of dimension three they maximize the functional area for variation keeping the boundary fixed. The subject has been treated under the viewpoint of variational methods, beginning with D.Brill and F.J.Flaherty [F-B] in 1975.

They considered 3-dimensional spacelike hypersurfaces in spacetimes and the critical points of their area functional. Almost at the same time and independently, Y. Choquet-Bruhat [C-B] obtained results on existence, uniqueness and non-existence of maximal hypersurfaces in Lorentz manifolds by using different methods. In 1983 S. Nishikawa [N] improved results of Choquet-Bruhat in [C-B] and also in [C-B-F-M]. With respect to the uniqueness of maximal spacelike hypersurfaces, S. Y. Cheng and S. T. Yau [C-Y] proved in 1976 that maximal entire graphs in the Lorentz-Minkowski space R_1^{n+1} are hyperplanes. This is a Bernstein-type property discovered by E. Calabi [C] in 1968 when he had obtained a solution for $n \leq 4$. Non-zero constant mean curvature spacelike hypersurfaces in certain Lorentz manifolds have a Bernstein-type property that will be discussed later in this work (see Section 5).

We are also leaving out existence results on constant and prescribed mean curvature hypersurfaces in Lorentz manifolds. Classical references are ([B-S], 1982) in the case of Minkowski space and ([Ge], 1983) in non-flat spacetimes. Recently, among other results, was proved in ([A-I], 1999) the existence of a complete constant mean curvature hypersurface in asymptotically flat Schwarzschild spacetimes satisfying a certain interior condition.

Stability questions for constant mean curvature compact hypersurfaces in Riemannian space forms began with Barbosa and do Carmo in ([B-C], 1984), and Barbosa , do Carmo and Eschenburg in ([B-C-E], 1988). They introduced the notion of stability and proved that spheres are the only stable critical points of the area functional keeping the volume constant. For spacelike hypersurfaces in Lorentz manifolds Barbosa and Oliker [B-O] in 1993, using the first variation formula for the functional area, proved that the constant mean curvature spacelike hypersurfaces are critical points for variation keeping the volume constant (see Section 4 for definitions). At this point the variational method for Riemannian and Lorentz manifolds coincide. They computed the second variation formula and obtained in the de Sitter space S_1^{n+1}that spheres maximize the area functional for variation keeping the volume constant; this fact determines the definition of stability which is given below in a more general form. It is also proved in [B-O] that if M^n is a complete immersed spacelike hypersurface in S_1^{n+1} with constant mean curvature H, then M^n is stable if M^n is compact, or if $H^2 \geq 1$, or $H^2 < 4(n-1)/n^2$. This theorem was generalized in [Br-C1] for constant $(r+1)$-mean curvature H_{r+1}. The r-mean curvature H_r is defined by

$$H_r = \frac{1}{\binom{n}{r}} S_r, \quad \text{with} \quad S_r = \sum_{i_1 < \cdots < i_r} k_{i_1} \cdots k_{i_r} \ ,$$

where $k_1, \ldots, k_n$ are the eigenvalues of the second fundamental form of M^n. Corresponding to the functional area for H,

$$A(t) = \int_M dM_t,$$

we have the functional A_r for H_{r+1} [B–C], given by

$$A_r(t) = \int_M F_r dM_t,$$

where

$$F_0 = 1, \ F_1 = S_1 \text{ and } F_r = S_r + \frac{(n-r+1)}{r-1} F_{r-2}, \text{ for } 2 \leq r \leq (n-1).$$

Here dM_t is the volume element of the metric on M^n induced by $X_t = X(t, .)$ with $X : (-\varepsilon, \varepsilon) \times M^n \to S_1^{n+1}$ a variation of M^n preserving volume. We say that M^n is r-stable if $A_r''(0) \leq 0$. Observe that 0-stable coincides with stable in the sense of Barbosa-Oliker, i.e, $A''(0) \leq 0$, for variation preserving volume (see Section 4 for details).

A natural step now is try to find conditions in more general spacetimes in order to guarantee that the constant mean curvature spacelike hypersurfaces are totally umbilical. In a series of papers, Alías, Romero and Sanchez ([A-R-S 1], 1995), ([A-R-S 2], 1997) and ([A-R-S 3], 1997), have considered Generalized Robertson-Walker (GRW) spacetimes. A GRW spacetime is a warped product $I \times M^n$ with Lorentzian metric

$$\overline{g} = -\pi_I^*(dt^2) + f(\pi_I)^2 \pi_M^*(g),$$

where π_I and π_M are the projections onto an open negative definite interval $(I, -dt^2)$ and a complete Riemannian manifold (M^n, g), respectively, and f is a positive differentiable function on M^n . (For $n = 3$ and M^n of constant sectional curvature this is a classical Robertson-Walker spacetime). Alías, Romero and Sanchez obtained a positive answer to Goddard's problem in the case of Einstein spatially closed GRW spacetime (see Theorem 5.3 in Section 5). They also succeeded in working in GRW spacetime satisfying the so called Time Convergence Condition (TCC), i.e, the Ricci tensor is non negative for every timelike vector; their main result (Theorem 5.1 in [A-R-S 1]) says :"In a GRW space satisfying TCC every constant mean curvature spacelike hypersurfaces is totally umbilical " (Theorem 5.6 in Section 5). They also treat a Bernstein-type property (see Theorem 5.8 in Section 5). In a recent paper Cao, Shen and Zhu [C-S-Z], using gradient estimates have obtained a Bernstein–type theorem for entire spacelike graphs with constant mean curvature in Lorentz-Minkowski spaces (see Theorem 5.9 in Section 5).

In [A-R-S 3] it is considered more general Lorentz manifolds which have a timelike conformal vector field (a GRW spacetime is an example of such spaces where of $f\dfrac{\partial}{\partial t}$ is a timelike conformal vector field) and similar results are proved (see Theorem 6.2 and 6.3 in Section 6).

Recently, S. Montiel ([M3], 1999) considered Lorentz manifolds that possess a closed conformal timelike vector field. Such a vector field determines a foliation of the manifold by leaves which are umbilical hypersurfaces and have constant mean curvature. Under the hypothesis known as null convergence condition (i.e, the Ricci tensor is non negative for every null vector), Montiel proved that compact spacelike immersed hypersurfaces of constant mean curvature in such manifolds are umbilical (Theorem 6.6 in Section 6, which generalizes Theorem 5.6 in Section 5). As a deeper result he obtained a classification of the umbilical spacelike immersed hypersurfaces of constant mean curvature in a Lorentz manifold that has a closed conformal timelike vector field and obeys the null convergence condition (see Theorem 6.7 in Section 6).

This work is an attempt to give a unified treatment of relevant facts and ideas on spacelike hypersurfaces of constant mean curvature in some Lorentz manifolds. It is a concise overview of the main results and methods in the field in the last twenty years.

2. Preliminaries

A Lorentz manifold is a pair (M, g), where M an $n-$dimensional $(n \geq 2)$ manifold a g is a metric tensor of index 1(i.e., g is a symmetric non-degenerate tensor field on M with index 1). We use an alternative notation for g writing $g(V, W) = \langle V, W \rangle$, for tangent vector fields V and W . The simplest example of Lorentz manifold is the Lorentz-Minkowski space $R_1^{n+1} = (R^{n+1}, g)$, where

$$g(v, w) = -v_0 w_0 + \sum_{i=1}^{n} v_i w_i,$$

with $v = (v_0, ..., v_n)$ and $w = (w_0, ..., w_n)$ in the $(n+1)-$dimensional Euclidean space R^{n+1}. Another example is the de Sitter space S_1^{n+1}, a hyperquadric in R_1^{n+2} given by $S_1^{n+1} = \{x \in R_1^{n+2}; < x, x >= 1\}$ which, with the induced metric, is a Lorentz manifold of constant sectional curvature 1.

Let M^n be an n-dimensional orientable Riemmanian manifold and $\overline{M}^{n+1}$ a Lorentz manifold. If $x : M^n \to \overline{M}^{n+1}$is an isometric immersion, define the second fundamental form of x with associated linear map B on the tangent space of M^n by

$$II(V, W) = \langle B(V), W \rangle = -\langle \nabla_V N\ , W \rangle,$$

where ∇ is the connection of $\overline{M}^{n+1}$ and N the unit vector field normal to $x(M^n)$.

If $e_1, ..., e_n$ is an orthonormal basis diagonalizing B, we denote by $k_1, ..., k_n$ the corresponding eigenvalues. We define the mean curvature of x by

$$H = \frac{1}{n} \sum_{i=1}^{n} k_i,$$

and the $r-$mean curvature of x by

$$H_r = \frac{1}{\binom{n}{r}} S_r, \quad \text{with } S_r = \sum_{i_1 < \cdots < i_r} k_{i_1} \cdots k_{i_r}.$$

Set $H_0 = S_0 = 1$ and $S_r = H_r = 0$ if $r \notin \{0, 1, \cdots, n\}$. Observe that $H_1 = H$ and H_2 is the (normalized) scalar curvature. Associated to B we define, inductively, the Newton transformations P_r by

$$P_0 = I \quad \text{and} \quad P_r = S_r I - BP_{r-1}, \quad 0 < r \leq n.$$

Associated to P_r we define a second order differential operator L_r by

$$L_r f = \operatorname{div}(P_r \nabla f),$$

where div and ∇ are, respectively, the divergent and the gradient. Observe that $L_0 = \Delta$, the Laplacian of M^n. One proves that if M is compact without boundary or if M is non–compact and f has compact support, then [B-C]

$$\int_M L_r(f) dM = 0 \qquad \text{and} \qquad \int_M f L_r(f) dM = -\int_M \langle P_r \nabla f, \nabla f \rangle dM.$$

For each r the ellipticity of L_r is equivalent to the positiveness of the quadratic form associated to P_r . But P_r and B are simultaneously diagonalizable. Hence, L_r is elliptic if and only if

$$\langle P_r e_i, e_i \rangle > 0, \ \forall i = 1, \cdots, n \quad ,$$

where $e_1, ..., e_n$ are the eigenvectors of B.

We say that a tangent vector v to a Lorentz manifold $\overline{M}^{n+1}$ is spacelike (timelike) if $v = 0$ or $\langle v, v \rangle > 0$ ($\langle v, v \rangle < 0$).

A submanifold M of a Lorentz manifold $\overline{M}^{n+1}$ is said to be spacelike if the metric on M induced by the metric of $\overline{M}^{n+1}$ is Riemannian ; that is, if every non-zero tangent vector to M^n is spacelike. An isometric immersion $x : M^n \to \overline{M}^{n+1}$ of a Riemannian manifold in a Lorentz manifold $\overline{M}^{n+1}$ is an example of a spacelike hypersurface in $\overline{M}^{n+1}$. We say that x(or $x(M^n)$) is totally umbilical (or just umbilical) if for a choice of an unit normal vector field N

$$II(V, V) = k_N < V, V >,$$

for some scalar function k_N on the domain of N, for every tangent vector field V.

If the mean curvature H of x is constant we define on the tangent spaces of M^n a linear map Φ by

$$\langle \Phi(V), W\rangle = H\langle V, W\rangle + \langle B(V), W\rangle,$$

for tangent vector fields $V$and W. It is easily checked that trace$(\Phi) = 0$ and that

$$\mid \Phi \mid^2 = \frac{1}{2n}\sum_{i<j}(k_i - k_j)^2,$$

where $k_1, \cdots, k_n$ are the eigenvalues of B. Thus, $\mid \Phi \mid^2 = 0$ if and only if $x(M^n)$ is totally umbilical.

3. Goddard's Conjecture: Solutions and Examples

In 1977 A. J. Goddard [G] conjectured that a complete spacelike hypersurface with constant mean curvature in the de Sitter space S_1^{n+1} must be umbilical.

It is known since 1981 a counterexample of Dajczer and Nomizu [D-N] of a complete surface of constant mean curvature $H > 1$ in S_1^3 which is not umbilical, given below.

Example 3.1. For $r > 0$ let $x:\ R^2 \to S_1^3$ be given by

$$x(\xi_1, \xi_2) = (r\cosh\frac{\xi_1}{r},\ r\sinh\frac{\xi_2}{r}, \sqrt{1+r^2}\cos\frac{\xi_2}{\sqrt{1+r^2}}, \sqrt{1+r^2}\sin\frac{\xi_2}{\sqrt{1+r^2}}).$$

This is a flat immersion with principal curvatures $\dfrac{r}{\sqrt{1+r^2}}$ and $\dfrac{\sqrt{1+r^2}}{r}$. Clearly x is not umbilical and $H > 1$.

The first partial solution of Goddard's conjecture was given by J. Ramanathan.

Theorem 3.2. ([R], 1987). *Let $x : M^2 \to S_1^3$ be a complete spacelike immersion of constant mean curvature H, with $H^2 < 1$. Then x is totally umbilical.*

Ramanathan's proof can not be extended to higher dimension as we see below.

Sketch of the proof:

Consider the quadratic differential (globally defined on M)

$$\Psi = (\frac{h_{11} - h_{22}}{2} - ih_{12})\lambda^2 dz^2,$$

where $z = x + iy$ are isothermal coordinates, λ is given by the metric $ds^2 = \lambda^2|dz|^2$ and h_{ij} are the coefficients of the second fundamental form of x; since H is constant, Ψ is holomorphic. One can prove easily that $H^2 < 1$ implies that M^2 is compact. Now, using that S_1^3 is diffeomorphic to $R \times S^2$ we prove that M^2 and S^2 are conformally diffeomorphic. Hence $\Psi \equiv 0$ and so x is totally umbilical.

Example 3.3. Let $r \geq 0$ and define $x_r : S^2 \to S_1^3$ by $x_r(\xi) = (\sinh r\xi, \cosh r\xi)$. The immersion x_r has constant mean curvature $H = \tanh(r)$. Thus, there exists a complete spacelike surface in S_1^3 with constant mean curvature H for every $H^2 < 1$ [Ra].

Theorem 3.4. ([A], 1987). *Let $M^{n+1}(c)$ be an $(n+1)$-dimensional Lorentz manifold of positive constant curvature c. Let M^n be a complete spacelike hypersurface immersed with constant mean curvature H in $M^{n+1}(c)$ such that*

(i) $|H| \leq c^{\frac{1}{2}}$ *if* $n = 2$;

(ii) $|H| < 2\,[(n-1)c]^{\frac{1}{2}}/\,n$, *if* $n > 2$.

Then M^n is totally umbilical.

Sketch of the proof:

First we prove that

$$\Delta(\,|B|^2 - nH^2\,) \geq K(n, c, |H|)(\,|B|^2 - nH^2\,)^2,$$

where $K(n, c, |H|)$ is a positive constant depending on n, c and $|H|$. On the other hand, from the hypotheses (i) and (ii) of the theorem, one shows that

M^n must have non negative Ricci curvature. Hence, by the generalized maximum principle of Cheng-Yau [C–Y], we conclude that $|B|^2 - nH^2 = 0$ on M^n and so M^n is totally umbilical, finishing the proof.

One year later, S. Montiel solved Goddard's conjecture for compact spacelike hypersurfaces in S_1^{n+1} without restriction over the range of H.

Theorem 3.5. ([M1], 1988). *Let M^n be a compact spacelike hypersurface immersed in the de Sitter space S_1^{n+1} with constant mean curvature H. Then it is totally umbilical. Moreover, up to a rigid motion in S_1^{n+1}, M^n coincides with some of the compact hypersurfaces described in the example below.*

Example 3.6. Define

$$M^n_{\sigma,\tau} = \{ \, x \in S_1^{n+1}; \langle \, x, a \, \rangle = \tau \, \},$$

where $a = (\, a_0, a_1, ..., a_{n+1}) \in R_1^{n+2}$, $| \, a \, |^2 = \sigma = 0, 1, -1$ and $\tau^2 > \sigma$. One proves that the second fundamental form B of $M^n_{\sigma,\tau}$ is given by

$$Bv = \frac{\tau}{(\tau^2 - \sigma)^{\frac{1}{2}}} v$$

for v tangent to $M^n_{\sigma,\tau}$. Hence, $M^n_{\sigma,\tau}$ is umbilical with mean curvature $H^2 = \dfrac{\tau^2}{(\tau^2 - \sigma)}$. It can be verified that $M^n_{\sigma,\tau}$ fall in the following three categories:

(a) if $\sigma = 1$, then $M^n_{1,\tau}$ is isometric to an n-dimensional hyperbolic space of constant sectional curvature $\dfrac{-1}{\tau^2 - 1}$ and H^2 ranges all possible values in $(1, \infty)$;

(b) if $\sigma = 0$, then $M^n_{0,\tau}$ is isometric to the Euclidean space R^n and $H^2 = 1$;

(c) if $\sigma = -1$, then $M^n_{-1,\tau}$ is isometric to an n-dimensional sphere with constant sectional curvature $\dfrac{1}{\tau^2 + 1}$ and H^2 takes all possible values in $[0, 1)$.

A characterization of the above examples is given by H is constant and $|\phi|^2 \equiv 0$ (see Preliminaries). The only case where $M^n_{\sigma,\tau}$ is compact is the last one, when $\sigma = -1$.

Remark 1. The above example describes all umbilical spacelike hypersurfaces in S_1^{n+1}. In fact, if M^n is umbilical, N the unit normal to M and λ the eigenvalue of the second fundamental form, take $a = \lambda i + N$, where $i : S_1^{n+1} \subset \mathbb{R}_1^{n+2}$ is the natural inclusion. Then $a \in \mathbb{R}_1^{n+2}$ is constant, $\mid a \mid^2 = \lambda^2 - 1$ and M^n is contained in the intersection of S_1^{n+1} and the hyperplane of $\mathbb{R}_1^{n+1}$ determined by a.

Remark 2. We observe that $M^n_{-1,\tau}$ can be described as a spacelike graph on S^n. In fact, by identifying the n-dimensional unit sphere with $\{x \in S_1^{n+1}; x_0 = 0\}$ we have a diffeomorfism $\Phi : R \times S^n \to S_1^{n+1}$ given by

$$\Phi(t,p) = \exp_p(tN_0) = \cosh tp + \sinh tN_0,$$

where exp is the exponential map of S_1^{n+1} and $N_0 = (1, 0, ..., 0)$ is the unit normal vector field of S^n with S^n viewed as a spacelike hypersurface of S_1^{n+1}. Now, given a function $u : S^n \to R$, we can construct an embedding $\psi : S^n \to S_1^{n+1}$defined by

$$\psi(p) = \phi(u(p), p) = \cosh u(p) + \sinh u(p) N_0.$$

One proves that

$$\langle \psi_*(v), \psi_*(w)\rangle = -\langle \nabla u, v\rangle\langle \nabla u, v\rangle + \cosh^2 u\langle v, w\rangle,$$

hence ψ is spacelike if and only if the function u satisfies

$$|\nabla u|^2 < \cosh^2 u.$$

We say that the embedding ψ is a spacelike graph on S^n corresponding to a function $u : S^n \to R$ with $|\nabla u|^2 < \cosh^2 u$. An important consequence is that "every spacelike hypersurface M^n in S_1^{n+1} is a spacelike graph (up to a diffeomorfism), provided it is compact". To see this, take $\pi \circ \phi^{-1} : M^n \to S^n$, where $\pi : R \times S^n \to S^n$ is the projection. Because M^n is compact, $\pi \circ \phi^{-1}$ is a diffeomorphism and so its inverse $(\pi \circ \phi^{-1})^{-1} : S^n \to S_1^{n+1}$ is an embedding which is a spacelike graph on S^n. In particular, $M_{-1,\tau}$ being compact is a spacelike graph on S^n, up to a diffeomorphism [M1].

Example 3.7. Let $\tau > 0$ and $1 \leq k \leq n-1$.Consider $M^n = \{x \in S_1^{n+1}; -x_0^2 + x_1^2 + ... + x_k^2 = -\sinh^2 \tau\}$ which is a spacelike embedded hypersurface in S_1^{n+1} isometric to the Riemannian product

$$H^k(1 - \coth^2 r) \times S^{n-k}(1 - \tanh^2 r)$$

of a k-dimensional hyperbolic space and an (n-k)-dimensional sphere, of constant sectional curvature $(1 - \coth^2 r)$ and $(1 - \tanh^2 r)$, respectively. Moreover, M^n is not umbilical and has constant mean curvature

$$H = \frac{1}{n}(k \coth r + (n-k) \tanh r).$$

Also one proves that

$$H^2 \geq \frac{4(n-1)}{n^2},$$

with equality when $k = 1$ and $\coth^2 r = n - 1$; hence $n > 2$. It can be proved that if $k = 1$ and $n > 2$, then H^2 takes all values in $[\frac{4(n-1)}{n^2}, \infty)$ [M1].

V. Oliker [O] obtained estimates of the normal curvature $k(p, v)$ of a hypersurface M^n in S^{n+1}, for $p \in M^n$ and $v \in T_pM$, in terms of the mean curvature H of M^n and its second covariant derivative $\nabla_{vv}H$. He considered the infimum C_1 and the supremum C_2 of

$$\left\{ H - \frac{\nabla_{vv}H}{1 - \frac{n^2H^2}{4(n-1)}} \right\}$$

taken over all points $p \in M^n$ and unit tangent vectors $v \in T_pM$.

Theorem 3.8. ([O], 1992). *Let M^n be a complete spacelike hypersurface immersed in S_1^{n+1} with mean curvature H and $H^2 < \frac{4(n-1)}{n^2}$. Then the following estimates hold for all $p \in M^n$ and $v \in T_pM$ with $|v| = 1$:*

(i) $C_1 \le k(p, v)$,

if $H \ge 0$ *and* $H - \nabla_{vv} H \ge 0$;

(ii) $k(p, n) \le C_2$,

if $H \le 0$ *and* $H - \nabla_{vv} H \le 0$.

To prove this theorem we first show that the hypothesis that $H^2 < \dfrac{4(n-1)}{n^2}$ implies that the Ricci curvature of M^n is bounded below by a positive constant. Consequently, by Bonnet-Myers theorem M^n is compact.

As a corollary, Theorem 3.8 gives, in particular, the above Theorem 3.4 (when $M^{n+1}(c) = S_1^{n+1}$) and (part of) Theorem 3.5.

Corollary 3.9. ([O]). *Let M^n be a complete spacelike hypersurface immersed in S_1^{n+1} with constant mean curvature H such that $H^2 < \dfrac{4(n-1)}{n^2}$. Then M^n is umbilical.*

Sketch of the proof:

Suppose $H \ge 0$. By (i) of Theorem 3.8, $k(p, v) \ge H$ for all $p \in M$ and any unit vector $v \in T_pM$. Consequently, $k(p, v) = H$, proving that M^n is umbilical in this case. If $H < 0$, the same conclusion follows from (ii) of Theorem 3.8, finishing the proof.

In Corollary 3.9, if M^n is not compact, the hypothesis that $H^2 < \dfrac{4(n-1)}{n^2}$ is essential to umbilicity of M^n(see Example 3.7).

Oliker's method consists in estimating the range of the variation of the principal curvature functions on a compact hypersurface M^n in terms of H and its derivative (up the second order). When H is constant this gives that M^n is umbilical. The estimates also gives the following stability property : if $H^2 < \dfrac{4(n-1)}{n^2}$ and H is near to a constant (in C^2 norm), then M^n is near to being umbilical. This is the content of the following

Theorem 3.10. ([O]). *Let M^n be a complete spacelike hypersurface immersed in S_1^{n+1} with mean curvature H with $H^2 < \dfrac{4(n-1)}{n^2}$. Suppose there exists constants H_0, $\delta \ge 0$ and $\varepsilon \ge 0$ such that* *(i)* $\varepsilon \leq H_0 - \delta \leq H \leq H_0 + \delta$;

(ii) $\sup |\nabla_{vv} H| < \varepsilon$,

where the supremum is taken over all points $p \in M^n$ *and* $v \in T_pM$, *with* $|v| = 1$. *Then the normal curvature* $k(p, v)$ *satisfies the inequality*

$$H_0 - d_1(\delta + \varepsilon) \leq k(p, v) \leq H_0 + d_2(\delta + \varepsilon),$$

where d_1*and* d_2 *are constants given by*

$$d_1 = \frac{1}{1 - \frac{n^2H^2}{4(n-1)}} \ \ and \ \ d_2 = (2n - 1)d_1.$$

Sketch of the proof:

From the hypothesis (i) and (ii) we obtain that $H \geq 0$ and $H - \nabla_{vv}H \geq H_0 - \delta - \varepsilon \geq 0$. Hence, by (i) of Theorem 3.8 we have that $C_1 \leq k(p, v)$. It follows that

$$H_0 - \delta - \frac{\varepsilon}{1 - \frac{n^2H^2}{4(n-1)}} \ \leq k(p, v). \qquad (*)$$

Since $d_1 > 1$, this yields

$$H_0 - d_1(\delta + \varepsilon) \leq \ k(p, v),$$

proving the inequality on the left hand side. The other inequality is proved also using (*) as follows: if $k_n(p)$ is the maximum of $k(p, v)$ at $p \in M^n$, then

$$\begin{array}{rcl} k(p, v) & \leq & k_n(p) = nH(p) - \sum_{j \neq n} k_j(p) \\ & \leq & nH_0 + n\delta - (n - 1)(H_0 - \delta - \varepsilon d_1) \\ & \leq & H_0 + (\delta + \varepsilon)d_{2,} \end{array}$$

finishing the proof.

In 1996 S. Montiel [M2] proved that the complete spacelike hypersurfaces with more than one topological end in S_1^{n+1} with $H^2 = \dfrac{4(n-1)}{n^2}$ are (up to rigid motions) the so called hyperbolic cylinders. These are isometric to the product $H^1(1-\coth^2 r)\times S^{n-1}(1-\tanh^2 r)$ of a hyperbolic line and a sphere of radii $\sinh r$ and $\cosh r$, respectively.

It would be natural to ask under which conditions a constant scalar curvature spacelike hypersurface M^n of S_1^{n+1} is totally umbilical. A result in this direction was obtained by the authors [Br–C2] under the following hypothesis on the mean curvature function H of M^n:

$$\sup H^2 \le \boldsymbol{C}_{\bar{R}},$$

where $\bar{R}=H_2-1$, with H_2 the normalized scalar curvature of M^n and

$$\mathcal{C}_{\bar{R}}=\frac{(n-1)\bar{R}}{n}+\frac{n(n-1)\bar{R}^2+4(n-1)\bar{R}+n}{n(n-2)(n\bar{R}+2)}.$$

Theorem 3.11. ([Br–C2], 1999). *Let M^n be an n-dimensional ($n\ge 3$) complete spacelike hypersurface immersed in S_1^{n+1} with constant normalized scalar curvature $H_2\ge 0$. Suppose the mean curvature function H satisfies the condition* $\sup H^2\le \boldsymbol{C}_{\bar{R}}$ *with $\bar{R}$ and $\boldsymbol{C}_{\bar{R}}$ as above. Then either*

(i) $\sup H^2=\bar{R}$, *and M^n is totally umbilical; or*

(ii) $\sup H^2=\boldsymbol{C}_{\bar{R}}$, *and M^n is isometric to $H^1(1-coth^2 r)\times S^{n-1}(1-tanh^2 r)$.*

The proof uses an expression of $L_1(nH)$ and the maximum principle at infinity of Omori and Yau (see [Br–C2]).

In [Z1], Y. Zheng gave an intrinsic condition for a compact spacelike hypersurface in S_1^{n+1} to be isometric to a sphere (improving a result in [Z2]).

4. The Variational Methods. Stability

Barbosa, do Carmo and Eschenburg studied constant mean curvature hypersurfaces in Riemannian manifolds of constant curvature as critical points of the variational problem of minimizing the area functional keeping the “balance of volume zero” (expression suggested in [B-C] for substituting the tradicionally used "volume constant"). They introduced the concept of stability and proved the following: "a constant mean curvature hypersurface is stable if and only if it is a geodesic sphere" ([B-C] and [B-C-E]).

For Lorentz manifolds, the application of variational methods to study hypersurfaces with constant mean curvature $H = 0$ has been considered, at least, since the work of Brill and Flaherty [F-B] in 1976. But for $H \neq 0$ it seems that it began with Barbosa and Oliker in [B-O]. In the sequel we discuss their work.

Let $\overline{M}^{n+1}$ be a Lorentz manifold and $x : M^n \to \overline{M}^{n+1}$ a spacelike immersion of a compact, connected and orientable manifold M^n with boundary (possibly empty) into $\overline{M}^{n+1}$.

Definition 4.1. *A variation of x is a differentiable map $X : M^n \times (-\varepsilon, \varepsilon) \to \overline{M}^{n+1}$ such that $X_t : M^n \to \overline{M}^{n+1}$ defined by $X_t(p) = X(p,t)$, $t \in (-\varepsilon, \varepsilon)$ and $p \in M^n$, is a spacelike immersion, $X_0 = x$ and $X_{t\,|\partial M} = x_{|\partial M}$, for every t. The area function $A : (-\varepsilon, \varepsilon) \to \mathbb{R}$ is defined by*

$$A(t) = \int_M dM_t,$$

where dM_t is the volume element of M in the metric induced by X_t. The volume function $V : (-\varepsilon, \varepsilon) \to R$ is defined by

$$V(t) = \int_{M \times [o,t]} X^* dM$$

with dM the volume element of $\overline{M}^{n+1}$.

Definition 4.2. *We say that x preserves the balance of volume if $V' \equiv 0$.*

Proposition 4.3. (1st variation formula). *Under the above notation*

$$A'(0) = -\int_M nHf dM,$$

and

$$V'(0) = \int_M f dM,$$

where $f = \langle \dfrac{\partial X}{\partial t}|_{t=0}, N \rangle$ and N is the unit normal field to $x(M^n)$.

Proposition 4.4. ([B-O], 1993). *Let $x : M^n \to \overline{M}^{n+1}$ be a spacelike immersion. Then x has constant mean curvature if and only if it is a critical point of the area for variation preserving the balance of volume.*

Thus, x has constant mean curvature if and only if $A'(0) = 0$ for all variations with $\int_M f dM = 0$.

Theorem 4.5. (2nd variation formula [B–O]). *Let $x : M^n \to \overline{M}^{n+1}$ be a spacelike immersion with constant mean curvature and X a variation of x preserving the balance of volume. Then $A''(0)(f)$ depends only on f and*

$$A''(0)(f) = \int_M (f\Delta f - |B|^2 f^2 + Ric(N) f^2) dM,$$

where $Ric(N)$ is the (normalized) Ricci curvature of $\overline{M}^{n+1}$ in the direction of the unit normal N.

Definition 4.6. *Let $x : M^n \to \overline{M}^{n+1}$ be a spacelike immersion with constant mean curvature. The immersion x is said to be stable if $A''(0) \leq 0$ for all variation preserving the balance of volume. If M is noncompact, we say that x is stable if for every compact submanifold $\tilde{M} \subset M$ with boundary, the restriction $x_{|_{\tilde{M}}}$ is stable.*

Example 4.7. The spheres of S_1^{n+1}. Let $M = \{x \in S_1^{n+1}; < x, a >= \tau\}$ where $a \in R_1^{n+2}$, with $|a| = -1$ and $\tau^2 > -1$, considered in Example 3.6 (c). For each τ, the mean curvature of M satisfies $H^2 = \dfrac{\tau^2}{\tau^2+1} \in [0, 1]$. One proves that M^n is isometric to an n-dimensional sphere with constant sectional curvature $\dfrac{1}{\tau^2+1}$ and we call them spheres of S_1^{n+1}. If $\tau = 0$, then $H = 0$ and M^n is totally geodesic.

Proposition 4.8. ([B-O]). *Spheres of S_1^{n+1} are stable. (This means that spheres in S_1^{n+1} maximize area for variation preserving the balance of volume).*

Sketch of the proof:

A sphere M^n of S_1^{n+1} is isometric to an Euclidean sphere of sectional curvature $\dfrac{1}{\tau^2+1}$; then the first eigenvalues of the Laplacian of M^n is $\lambda_1 = \dfrac{n}{\tau^2+1}$. On the other hand, the mean curvature H of M^n satisfies $H^2 = \dfrac{\tau^2}{\tau^2+1}$. Since $Ric(N) = n$ in S_1^{n+1}, the second variation formula yields

$$\begin{aligned} A''(0)(f) &= \textstyle\int_M (f\Delta f - \mid B \mid^2 f^2 + nf^2)dM \\ &= -\textstyle\int_M (|\nabla f|^2 + (|B|^2 - n)f^2 dM \\ &\leq -\textstyle\int_M (\lambda_1 + nH^2 - n)f^2 dM \leq 0, \end{aligned}$$

proving the proposition.

Theorem 4.9. ([B-O]). *Let $x : M^n \to S_1^{n+1}$ be a spacelike immersion with H constant of a complete, connected and orientable manifold. Then x is stable if one of the following conditions holds:*

(a) M^n is compact ; (b) $H^2 \geq 1$; (c) $H^2 < \dfrac{4(\ n-1)}{n^2}$.

Sketch of the proof:

(a) If M^n is compact, the hypothesis of H constant implies that x is umbilical, by Theorem 3.5. Hence $x(M)$ is a sphere of S_1^{n+1}(Example 3.6 (c) and Remark 1 in the previous section). We have seen in Proposition 4.8 that spheres are stable and so (a) is proved. (b) Suppose $H^2 \geq 1$. Then $|B|^2 \geq nH^2 \geq n$. By using this in the formula of the second variation we get $A''(0) \leq 0$ and so x is stable. (c) If $H^2 < \dfrac{4(n-1)}{n^2}$ one shows that

$$Ric(v) \geq [(n-1) - \frac{1}{4}(nH)^2]|v|^2,$$

for any tangent vector to M^n with

$$[(n-1) - \frac{1}{4}(nH)^2] > 0.$$

By Bonnet-Myers theorem M^n is compact, hence x is stable by (a).

Corollary 4.10. *Let $x : M^2 \to S_1^3$ be a constant mean curvature spacelike immersion of a complete, connected and orientable surface M^2. Then x is stable.*

The above results in [B-O] have been generalized to constant $(r+1)-$mean curvature H_{r+1} (see definition in Section 2) by Brasil Jr and Colares in [Br-C1]. They also proved that compact spacelike hypersurfaces with positive constant $(r+1)-$mean curvature in S_1^{n+1} are umbilic.

Theorem 4.11. ([Br-C1], 1998). *Let $x : M^n \to S_1^{n+1}$ be a spacelike immersion of a compact, connected and orientable Riemannian manifold M^n. If x has constant $(r+1)-$mean curvature $H_{r+1} > 0$, for some $0 \leq r \leq n-1$, then x is umbilical.*

The proof is a computation using the inequality $H_1 H_{r+1} \geq H_{r+2}$ and combining the expressions of $L_r(\langle x, A\rangle)$ and $L_r(\langle N, A\rangle)$, for a fixed vector A and unit normal N [Br-C1] . Independently, an equivalent result was proved in [A-A-R]. For $r = 1$ there are proofs in [L], [C-I], [A-A-R] and [M3].

The concept of $r-$stability of Barbosa-Colares [B-C] is introduced in S_1^{n+1}as follows.

Let $x : M^n \to S_1^{n+1}$ be a spacelike immersion of a compact, connected and orientable manifold M^n with boundary (possibly empty) and $X : M^n \times (-\varepsilon, \varepsilon) \to S_1^{n+1}$ a variation of x with f of compact support.

Define, for each r, $0 \leq r \leq n-1$ and $t \in (-\epsilon, \epsilon)$, the functional

$$A_r(t) = \int_M F_r(S_1, ..., S_r) dM_t,$$

where the functions F_r are given by

$$F_0 = 1, \ F_1 = S_1$$

and

$$F_r = S_r + \frac{n-r+1}{r-1} F_{r-2}, \text{ for } 2 \leq r \leq n-1.$$

Here $S_r = \binom{n}{r} H_r$.

As in [B-C] it can be proved that the critical points of A_r for variation preserving the balance of volume are hypersurfaces of constant $(r+1)-$mean curvature H_{r+1}(see [Br-C1]).

Proposition 4.12. (2nd Variation Formula [Br-C1]) *Let $x : M^n \to S_1^{n+1}$ be a spacelike immersion with constant $(r+1)-$mean curvature H_{r+1} and X a variation preserving the balance of volume. Then $A''(0)(f)$ depends only on f and*

$$A_r''(0)f = (r+1)\int_M f[L_r f - (S_1 S_{r+1} - (r+2)S_{r+2})f + (n-r)S_r f]dM,$$

where L_r is the operator defined in Section 2.

Definition 4.13. *Let $x : M^n \to S_1^{n+1}$ be a spacelike immersion of a connected and orientable manifold M^n with constant $(r+1)-$mean curvature H_{r+1}. We say that x is r-stable if $A_r''(0)f \leq 0$ for all variation preserving the balance of volume. If M^n is not compact we say that x is $r-$stable if for every compact submanifold $\tilde{M} \subset M$ with boundary, the restriction $x_{|\tilde{M}}$ is r-stable.*

Theorem 4.14. ([Br-C1]). *Spheres of S_1^{n+1} are r-stable.*

Theorem 4.15. ([Br-C1]). *Let $x : M^n \to S_1^{n+1}$ be a spacelike immersion of a complete, connected and orientable manifold M^n with constant $(r+1)-$mean curvature $H_{r+1} > 0$. Then x is r-stable if one of the following conditions holds:*

(i) M^n is compact ;

(ii) the square of the mean curvature $H_1^2 \geq 1$;

(iii) $H_1^2 < \dfrac{4(n-1)}{n^2}$.

Sketch of the proof :

Suppose M^n is compact. By the Theorem 4.11 above, x is umbilical and by Example 3.6 (c) and Remark 1 in the previous section, $x(M^n)$ is a sphere. By Theorem 4.14 spheres are $r-$stable. Suppose now the condition (ii) holds . Because $H_{r+1} > 0$, one proves that L_r is elliptic (see Proposition 3.2 in [B-C]). Therefore,

$$\langle P_r(e_i), e_i \rangle > 0,$$

where $e_1, ..., e_n$ are the eigenvectors corresponding to $k_1, ..., k_n$. Thus,

$$\langle P_r \nabla f, \nabla f \rangle > 0.$$

On the other hand, the hypothesis $H_1 \geq 1$ implies that

$$S_1 S_{r+1} - (r+2) S_{r+2} - (n-r) S_r \geq 0.$$

Therefore, for any compact submanifold $\tilde{M} \subset M$ with boundary

$$A''(0)(f) = -(r+1) \int_{\tilde{M}} [\langle P_r \nabla f, \nabla f \rangle + [(S_1 S_{r+1} - (r+2) S_{r+2}) - (n-r) S_r] f^2 \leq 0,$$

giving that x is $r-$stable in this case.

Suppose now that $H^2 < \dfrac{4(n-1)}{n^2}$. As in [B-O] , one proves that

$$Ric(v) \geq [(n-1) - \frac{1}{4}(nH)^2]|v|^2,$$

with $(n-1) - \frac{1}{4}(nH)^2 > 0$. By Bonnet-Myers theorem M^n is compact, hence by (i) x is r-stable.

5. Goddard's Problem in Generalized Robertson-Walker spacetimes. A Bernstein-Type Property

Following [A-R-S 2] we consider the product manifold $\overline{M} = I \times M^n$ with the Lorentzian metric

$$\overline{g} = -\pi_I^*(dt^2) + f(\pi_I)^2 \pi_M^*(g),$$

where π_I and π_M denote the projections onto I and M^n, respectively. Here (M^n, g) is a connected complete Riemannian manifold with dimension $n \geq 2$, $f : I \to \mathbb{R}$ is a positive differentiable function and I is an open interval of $\mathbb{R}$. Then $(\overline{M}, \overline{g})$ is a warped product with base $(I, -dt^2)$, fibre (M, g) and warping function f. The product $(\overline{M}, \overline{g})$ is called Generalized Robertson–Walker spacetime (GRW spacetime). When $n = 3$ and M^n has constant sectional curvature, then $(\overline{M}, \overline{g})$ is a classical Robertson–Walker spacetime (see [ON], for example).

Example 5.1. When M is the Euclidean space $\mathbb{R}^{n+1}, I = \mathbb{R}$ and f is the identity, then $\overline{M}$ is the Lorentz–Minkowski space $\mathbb{L}^{n+2}$.

Example 5.2. In $\overline{M} = I \times M^n$, let M^n be the sphere S^n, $I = \mathbb{R}$ and $f(t) = \cosh(t)$. Then $\overline{M}$ is the de Sitter space S_1^{n+1}.

We denote $\partial_t = \dfrac{\partial}{\partial t}$ the timelike vector field globally defined on $\overline{M}$ which determines a time-orientation on $\overline{M}$. If S is a spacelike hypersurface in $\overline{M}$ we have a unit globally defined timelike normal vector field N on S whose time-orientation coincides with that of ∂_t.

In 1995 was formulated in [A-R-S 1] the following Goddard–type question: "When is a complete spacelike hypersurface of constant mean curvature in a GRW spacetime totally umbilical or a slice?"

To study this question we first consider Einstein GRW spacetimes.

We say that $\overline{M} = I \times M^n$ is Einstein GRW space if the Ricci tensor of $\overline{M}$ satisfies $\overline{R}ic = c\overline{g}$, for some constant c. For spatially closed Einstein GRW spaces (i.e., when M^n is compact without boundary), the above question has an answer.

Theorem 5.3. ([A-R-S 2], 1997). *Every compact spacelike hypersurface of constant mean curvature in a spatially closed Einstein GRW spacetime is totally umbilical.*

Sketch of the proof:

Let S be a spacelike hypersurface with mean curvature H in $\overline{M}$. Consider the timelike vector field $Z = f(\pi)\partial_t$ on $\overline{M}$. For the component tangent to S, $Z^{\top}$, we have

$$div\left(Z^{\top}\right) = nf'(\pi) + nH\overline{g}(Z, N),$$

where N is normal to S. Using the Codazzi equation to compute $div(BZ^T)$ and integrating we get

$$\int_S \{(n-1)\overline{g}(\nabla H, Z) + \overline{R}ic(Z^{\top}, N) + \left(\mid B \mid^2 - nH^2\right)\overline{g}(Z, N)\}dS = 0, \quad (5.1)$$

where ∇H denotes the gradient of H and B the second fundamental form of S. Hence, because H is constant and $\overline{M}$ is Einstein, (5.1) yields

$$\int_S \left(|B|^2 - nH^2\right)\overline{g}(Z, N)dS = 0.$$

Since $|B|^2 - nH^2 \geq 0$, $f(\pi) > 0$ and N is normal to S, we must have $\mid B \mid^2 = nH^2$ and so S is umbilical, finishing the proof.

The above proof gives an idea of one of the methods used by Alias, Romero and Sanchez: integral formulas obtained from derivation of the timelike vector field $Z = f(\pi)\partial_t$ on $\overline{M}$.

Example 5.4. In a GRW spacetime $\overline{M} = I \times M^n$, the hypersurface $\{t\} \times M^n$ is an example of a spacelike slice. Following Alias, Romero and Sanchez we define a spacelike slice as an immersion $x : S \to \overline{M}$ such that $\pi_I \circ x$ =constant. These are umbilical spacelike hypersurface of constant mean curvature $H = \dfrac{f'(t)}{f(t)}$.

Theorem 5.5. ([A-R-S 2]). *Let $x : S \to \overline{M}$ be a spacelike immersion of a compact manifold with non–positive definite Ricci tensor in a spatially closed Einstein GRW spacetime. If x has constant mean curvature, then it is a spacelike slice.*

The proof uses Theorem 5.3 and the fact of that $Z^\top = \left(f(\pi)\partial_t\right)^\top$ is a Killing vector field which is then parallel. This implies that $\pi_I \circ x$ is constant, hence x is a spacelike slice, proving Theorem 5.5.

If a GRW spacetime $\overline{M}$ satisfies $\overline{R}ic(W, W) \geq 0$ for every timelike vector field W we say that $\overline{M}$ satisfies a *Timelike Convergence Condition* (TCC).

Theorem 5.6. ([A-R-S 1], 1995). *Let $\overline{M}$ be a GRW space satisfying TCC. Then every compact spacelike hypersurface S of constant mean curvature is totally umbilical.*

Sketch of the proof:

By using the formula (5.1) we get

$$\int_S \left\{\overline{R}ic(Z^\top, N) + \left(\mid B \mid^2 - nH^2\right)\overline{g}(Z, N)\right\} dS = 0,$$

since H = constant. We then can prove that we must have $\mid B \mid^2 - nH^2 = 0$ on S and so S is umbilical, finishing the proof.

Remark 1. ([A-R-S 1]) In $\overline{M}$, TCC holds if and only if

$$f'' \leq 0$$

and

$$Ric \geq (n-1)\left(ff'' - (f')^2\right),$$

where Ric is the Ricci tensor of M^n. Let $S \subset \overline{M}$ be a compact spacelike hypersurface and N the unit vector field normal to S. Then

$$\overline{Ric}\left(Z^{\top}, N\right) = 0 \Leftrightarrow Ric(\widetilde{N}, \widetilde{N}) = (n-1)\left(ff'' - (f')^2\right)\overline{g}(\widetilde{N}, \widetilde{N}),$$

where $\widetilde{N}$ is the projection $d\pi_M(N)$ of N on M^n.

Theorem 5.7. ([A-R-S 1]). *Let $\overline{M}$ be a GRW spacetime satisfying TCC with $Ric > (n-1)(ff'' - (f')^2)$. Then every compact spacelike hypersurface S of constant mean curvature in $\overline{M}$ is a spacelike slice.*

Sketch of the proof:

By Theorem 5.6 above, S is totally umbilical. On the other hand, Remark 1 and our hypothesis on Ric yield $\widetilde{N} = 0$. Therefore, $\pi_I(S) =$ constant and so S is a spacelike slice, proving the theorem.

We will see that this theorem gives a solution of a Bernstein–type problem for constant mean curvature spacelike graphs in $\overline{M}$ (see Theorem 5.8).

Let $\overline{M} = I \times M^n$ a GRW spacetime with M^n complete and warping function $f : I \to \mathbb{R}$. We have seen that the metric $\overline{g}$ of $\overline{M}$ is given by

$$\overline{g} = -\pi_I^*(dt^2) + f(\pi_I)^2 \pi_M^*(g),$$

where g is the Riemannian metric in M^n. Now let $u : M^n \to I$ be a differentiable function and consider the graph

$$M_u = \{(u(p), p); p \in M^n\} \subset \overline{M}.$$

Then, the metric g_u on M^n induced from the metric $\overline{g}$ via M_u is given by

$$g_u = -du^2 + f(u)^2 g,$$

which is positive definite if and only if $\mid \nabla u \mid < f(u)$ everywhere on M^n. Following ([A-R-S 3]), consider the functional $A(u)$ defined by

$$A(u) = \int_M (f(u))^{n-1} \sqrt{(f(u))^2 - \mid \nabla u \mid^2} dM, \tag{5.2}$$

where dM is the volume element of (M, g). Impose the constraint

$$\int_M \left(\int_{t_o}^u (f(t))^n \, dt \right) dM = \text{constant}. \tag{5.3}$$

Then, for critical points of such a functional, the Euler–Lagrange equation is written as follows

$$div \left(\frac{\nabla u}{f(u)\sqrt{f(u)^2 - \mid \nabla u \mid^2}} \right) = nH - \frac{f'(u)}{\sqrt{f(u)^2 - \mid \nabla u \mid^2}} \left(u + \frac{\mid \nabla u \mid^2}{f(u)^2} \right) \tag{5.4}$$

with

$$|\nabla u| < f(u),$$

where H is a real number.

Observe that $A(u)$ is the area of (M, g_u) and (5.3) the volume constraint.

In [A-R-S 3] is also pointed out the particular cases considered below.

(a) If $M = R^n$, $I = R$ and $f = 1$, then $M = R_1^{n+1}$; in this case, for $H = 0$ equation (5.4) has the Bernstein–type property: “The entire solutions of (5.4) are affine functions” [C-Y]. For $H \neq 0$, there is no uniqueness: many solutions were found by S. Stumbles [S] and A. Treibergs [T].

(b) If $M^n = S^n$, $I = \mathbb{R}$ and $f(t) = \cosh(t)$, then $\overline{M} = S_1^{n+1}$ and all solutions of (5.4) were found by S. Montiel in [M1]. Montiel proved that for $H^2 < 1$ equation (5.4) has a Bernstein–type property.

It seems that in this direction the most general uniqueness results is given by the theorem below. It is a reformulation of Theorem 5.7 for graphs showing that some graphs in a GRW spacetime $\overline{M} = I \times M^n$ over M^n exhibit a Bernstein-type property.

Theorem 5.8. ([A-R-S 3]). *Let (M, g) be a compact Riemannian manifold, H a real number and $f : I \to \mathbb{R}$ a positive differentiable function. Suppose $f'' \leq 0$ and $Ric > (n-1)Sup(ff'' - (f')^2)$ or $f'' < 0$ and (M, g) is non-negative Ricci curved. Then the only solutions to the constant mean curvature equation (5.4) are the constant functions.*

Recently, Cao, Shen and Zhu ([C–C–Z]) obtained a Bernstein–type theorem for constant mean curvature spacelike hypersurface M^n in the Lorentz–Minkowski space $\mathbb{R}_1^{n+1}$ in terms of the Gauss map of M into the $n-$dimensional hyperbolic space $\mathbb{H}^n(-1)$.

If N is a unit vector field normal to M^n identified as a point of $\mathbb{H}^n(-1)$, the Gauss map $G : M^n \to \mathbb{H}^n(-1)$ is defined by $G(p) = N(p),\ p \in M^n$.

Theorem 5.9. ([C–S–Z], 1998) *Let M^n be a spacelike hypersurface of constant mean curvature in $\mathbb{R}_1^{n+1}$. Let g be a function on M^n such that e^{-g} is a superharmonic.If the image of the Gauss map $G : M^n \to \mathbb{H}^n(-1)$ lies in $\{(\gamma_1, \cdots .\gamma_n); \gamma_n > e^g\}$, then M^n is a hyperplane. In particular, this is the case if the Gauss image is contained in the half–space $\{\gamma_1 > c\}$ for some constant c.*

The proof uses a sharp gradient estimate for the Gauss map, obtained in [C-S-Z]. First they show that constant mean curvature of M^n is equivalent to the harmonicity of the Gauss map.

6. Umbilical Spacelike Hypersurfaces in More General Spacetimes

One of the methods used by Alías et al. to prove the results in a GRW spacetime $\overline{M}$ (Theorems 5.3, 5.5, 5.6 and 5.7 in Section 5) uses integral formulas obtained from derivation of the timelike vector $Z = f(\pi)\partial_t$ globally defined on $\overline{M}$. Formula (5.1), for example, was obtained by computing $div(BZ^T)$ and integrating the expression obtained over the hypersurface S.

In fact, a spacetime having a closed conformal timelike vector field (see Definition below) is a natural universe to derive Minkowski-type integral formulas.

A Lorentz manifold $(\overline{M}, \overline{g})$ which admits a timelike conformal vector field (i.e., a vector field Z on $\overline{M}$ such that $\mathcal{L}_Z\overline{g} = 2\varrho\overline{g}$, where $\mathcal{L}_Z$ is the Lie derivation with respect to Z and ρ is a real function) is called a *conformally stationary spacetime* (CS spacetime) [A-R-S 3].

Example 6.1. Let $\overline{M}$ be a GRW spacetime. Then $Z = f(\pi)\partial_t$ is a conformal timelike vector field. In fact, if $\mathcal{L}_Z$ is the Lie derivation with respect to Z on $\overline{M}$, then $\mathcal{L}_Z\overline{g} = f'(\pi)\overline{g}$, where $\overline{g}$ is the metric of $\overline{M}$.

Every spacelike hypersurface S in a CS spacetime $\overline{M}$ has a unit normal vector field N satisfying

$$\overline{g}(Z, N) \geq (-\overline{g}(Z, Z))^{\frac{1}{2}} > 0.$$

For a constant mean curvature spacelike hypersurface S in a CS spacetime, the same formula in (5.1) holds and we then obtain :

$$\int_S \left\{\overline{R}ic(Z^\top, N) + \overline{g}(Z, N)\left(\mid B\mid^2 - nH^2\right)\right\} dS = 0, \tag{6.1}$$

where $\overline{R}ic$ is the Ricci tensor of $\overline{M}$, B and H the second fundamental form and the mean curvature of S, respectively and Z^T is the component of the conformal field Z tangent to S.

By using (6.1) we obtain the following

Theorem 6.2. ([A-R-S 3], 1997). *If a CS spacetime $\overline{M}$ is Einstein, then every compact spacelike hypersurface of constant mean curvature in $\overline{M}$ is totally umbilical.*

We have seen that if $(\overline{M}, \overline{g})$ is a GRW spacetime with warping function f, then $Z = f(\pi)\partial_t$ is a conformal timelike vector field with $\mathcal{L}_Z\overline{g} = f'(\pi)\overline{g}$. For a CS spacetime $\overline{M}$ we have $\mathcal{L}_Z\overline{g} = 2\varrho\overline{g}$, where ϱ is a differentiable function on $\overline{M}$.

Theorem 6.3. ([A-R-S 3]) *Let $\overline{M}$ be a CS spacetime with ϱ = constant. If $\overline{M}$ obeys TCC, then every compact spacelike hypersurface S of constant mean curvature is totally umbilical.*

Sketch of the proof:

We have (by formula (2.3) in [A-R-S 3])

$$div\left(\left(\overline{\nabla}_N Z\right)^{\top}\right) = n\overline{g}\left(\overline{\nabla}\varrho, N\right) + \overline{R}ic(Z,N),$$

hence

$$\int_S \left\{n\overline{g}\left(\overline{\nabla}\varrho, N\right) + \overline{R}ic(Z,N)\right\} dS = 0. \tag{6.2}$$

Since ϱ =constant we obtain

$$\int_S \overline{R}ic(Z,N)dS = 0.$$

Using this in (6.1) we get

$$\int_S \overline{g}(Z,N)\{\overline{R}ic(N,N) + \left(| B |^2 - nH^2\right)\}dS = 0.$$

Since $\overline{M}$ satisfies TCC we have that $\overline{R}ic(N,N) = 0$. Hence we must have $| B |^2 = nH^2$ and so S is umbilical, finishing the proof.

Recently, S. Montiel [M3] considered a Lorentz manifold $\overline{M}^{n+1}$ having a closed conformal timelike vector field Z, i.e., Z satisfying

$$\nabla_v Z = \phi v, \text{ for every } v \in T\overline{M}^{n+1},$$

where ∇ is the Levi-Civita connection and ϕ is a differentiable real function on $\overline{M}^{n+1}$. Clearly, a GRW spacetime is an example of such a manifold. He first proved the following

Proposition 6.4. *Let $(\overline{M}^{n+1}, \overline{g})$ be a Lorentz manifold possessing a closed conformal timelike vector field Z. Then* *a) the n-dimensional distribution D defined on $\overline{M}^{\,n+1}$ by*

$$p \in M^{n+1} \to D(p) = \{v \in T_p\overline{M}^{n+1};\ \overline{g}(Z(p), v) = 0\}$$

determines a codimension one spacelike foliation $\mathcal{F}(Z)$ which is oriented by Z ; moreover, the functions $\langle Z, Z\rangle$, $divZ$ and ϕZ are constant on connected leaves of $\mathcal{F}(Z)$;

b) the unit timelike vector field $Z_0 = \dfrac{Z}{(-\overline{g}(Z,Z))^{\frac{1}{2}}}$ on $\overline{M}^{\,n+1}$ satisfies

$$\nabla_{Z_0}Z_0 = 0 \ \text{ and } \ \nabla_v Z_o = \frac{\phi}{(-\overline{g}(Z_0,Z_0))^{\frac{1}{2}}}v \ \text{ if } \ \overline{g}(v, Z_0) = 0.$$

The condition b) implies that the flow of Z_o is a unit speed timelike geodesic flow mapping homothetically leaves of $\mathcal{F}(Z)$ on leaves of $\mathcal{F}(Z)$ and each leaf of $\mathcal{F}(Z)$ is umbilical and has constant mean curvature given by $-\dfrac{\phi}{(-\overline{g}(Z_0,Z_0))^{\frac{1}{2}}}$.

Example 6.5. In the de Sitter space $S_1^{n+1} \subset R^{n+2}$ defined in Section 2 consider the timelike vector field

$$Z(p) = v - \overline{g}(p, v)p,\ p \in S_1^{n+1}$$

for a chosen timelike vector $v \in R^{n+2}$, with $|v|^2 = -1$. It foliates S_1^{n+1} by means of umbilical round spheres, given by $g(p, v) = \tau,\ \tau \in R$, with radii $\sqrt{1+\tau^2}$(see Example 3.6 in Section 3).

On a Lorentz manifold possessing a closed conformal timelike vector field, S. Montiel considered the so called *null convergence condition* given by

$$\overline{Ric}(v, v) \geq 0,\ \text{for each } v \in T\overline{M},\ |v|^2 = 0,$$

which is weaker then TCC. He proved the following theorem which generalizes Theorem 5.6 of the previous section.

Theorem 6.6. ([M3], 1999). *Let $\overline{M}^{n+1}$ be a Lorentz manifold having a closed conformal timelike vector field and obeying the null convergence condition. Then* *any compact spacelike hypersurface S immersed in $\overline{M}^{n+1}$ with constant mean curvature is umbilical.*

Sketch of the proof:

Choose the unit vector field N on S to have $\overline{g}(Z,N)\geq 1$. Denote $Z_0=\dfrac{Z}{\overline{g}(Z,Z)^{\frac{1}{2}}}$. Now multiplying the first Minkowski formula by the constant mean curvature H of S to obtain

$$\int_M\{H\phi - H^2\overline{g}(Z,N)\}dS=0, \tag{6.3}$$

where ϕ is given by $\nabla_v Z=\phi v,\ v\in TM$. On the other hand, the second Minkowski formula ([M3], p. 542) read as follows :

$$\int_S\{n(n-1)[H\phi - H_2\overline{g}(Z,N)]+[\overline{R}ic(N,N)-\overline{R}ic(Z_0,Z_0)]\overline{g}(Z,N)\}dS=0, \tag{6.4}$$

where H_2 is the normalized second elementary symmetric polynomial in the principal curvatures of S.

Combining (6.3) and (6.4), we obtain

$$\int_S\{n(n-1)[H^2-H_2]+\overline{R}ic(N,N)-Ric(Z_0,Z_0)]\overline{g}(Z,N)\}dS. \tag{6.5}$$

By Lemma 3 in [M3], under the hypotheses on $\overline{M}^{n+1}$, we have

$$Ric(v,v)\geq Ric(Z_0,Z_0),\ \text{for each } v\in T\overline{M}^{n+1}, |v|=-1. \tag{6.6}$$

By (6.6), $Ric(N,N)\geq Ric(Z_0,Z_0)$. On the other hand, from Schwarz inequality we have $H_2\leq H^2$, holding the equality at umbilical points of S. Since $\overline{g}(Z,N)\geq 1$, (6.5) then gives that S must be totally umbilical, finishing the proof.

Montiel also obtained a deeper result : a classification of the umbilical spacelike hypersurfaces with constant mean curvature in a Lorentz manifold possessing a closed conformal timelike vector field and obeying the null convergence condition.

Theorem 6.7. ([M3]). *Let $\overline{M}^{n+1}$ be a Lorentz manifold possessing a closed conformal timelike vector field Z and obeying the null convergence condition. If S is a compact constant mean curvature umbilical immersed spacelike hypersurface of $\overline{M}$, then one of the following statements holds:*

(i) *S is a leaf of the foliation $\mathcal{F}(Z)$;*

(ii) *$\overline{M}^{n+1}$ is, locally, a de Sitter space and S is a round hypersphere described in Example 3.6 of the Section 3 ;*

(iii) *locally, $\overline{M}^{n+1}$ is a semi-Riemannian product $R_1^2 \times Q^{n-1}$ and S is locally a product $R \times Q^{n-1}$ immersed as $\gamma \times I_{Q^{n-1}}$, where γ is a spacelike line in the Lorentz plane $R^2_{1.}$*

Sketch of the proof:

Since S is umbilical and has constant mean curvature we have that

$$\overline{R}ic(N,N) = Ric(Z_0, Z_0),$$

where N is the unit normal vector field on S and $Z_0 = \dfrac{Z}{\overline{g}(Z,Z)^{\frac{1}{2}}}$.

Suppose S is not a leaf of $\mathcal{F}(Z)$. Then Z^T (the tangent component of Z) is a non trivial vector field on S. On the other hand, the umbilicity of S yields

$$\nabla^S_v Z^T = (\phi - H\overline{g}(Z,N))v, \ v \in TS,$$

where ∇^S is the covariant derivative and H the mean curvature of S. Hence, Z^T is a closed conformal vector field on the Riemannian manifold S. Therefore, from Theorem 3 of [ST], S is diffeomorphic to either a n-sphere or a twisted n-torus, and Z vanishes at most at two points of S that we denote by a and b.

Since $Z^T = Z + \langle Z, N\rangle N$, by using (6.6) we conclude that on the open set $S - \{a,b\}$ the Ricci curvature of M attains the least value in the Lorentz plane spanned by Z and N. Thus, we can obtain a unit spacelike vector field U on $S - \{a,b\}$ such that U is tangent to $\overline{M}^{n+1}$, orthogonal to Z_0 and attains the equality

$$\overline{R}ic(U,U) = -\overline{R}ic(Z_0, Z_0).$$

Now, we use that

$$\overline{g}(Z,Z)\overline{Ric}(U,U) = (n-1)(\phi^2 - X\phi)$$

and that the tensor Ric is invariant under the flow F_t of Z_0 to obtain, at $p \in S - \{a,b\}$,

$$\begin{aligned}(n-1)(\phi^2 - Z\phi)(p) &= \overline{g}(Z,Z)\overline{Ric}((dF_t)(U_p),(dF_t)(U_p)) \\ &\geq -\overline{g}(Z,Z)\overline{g}((dF_t)_p(U_p),(dF_t)_p(U_p)). \\ &\quad \overline{Ric}((Z_0)_{F_t(p)},(Z_0)_{F_t(p)}).\end{aligned}$$

Here the inequality comes from (6.6). We then finally obtain

$$(\phi^2 - Z\phi)(p) \geq (\phi^2 - Z\phi)(F_t(p)),$$

for any $p \in S - \{a,b\}$ and any t. Thus, since $Z\phi$ is locally constant on the leaves $\mathcal{F}(Z)$ (Prop. 6.4, (a)), the function $(\phi^2 - Z\phi)$ is constant on the union Ω of the leaves of $\mathcal{F}(Z)$ intersecting $S - \{a,b\}$ which is a connected open set of $\overline{M}^{n+1}$.

Therefore, by using the characterization of Z we finally obtain

$$Z\left(\frac{Z\phi}{\overline{g}(Z,Z)}\right) = \frac{Z(Z\phi - \phi^2)}{\overline{g}(Z,Z)} = 0$$

on Ω and so

$$\frac{X\phi}{\overline{g}(Z,Z)} = -c, \tag{6.7}$$

for some $c \in R$. Since $\nabla\phi = \dfrac{Z\phi}{\overline{g}(Z,Z)}Z$, we then have that $\nabla\phi = -cX$ on Ω. This together with the umbilicity of S yield

$$\nabla_\Sigma\phi = -cZ^T \quad \text{and} \quad \nabla_\Sigma\overline{g}(Z,N) = -HZ^T. \tag{6.8}$$

Deriving once more we get

$$\nabla^2_\Sigma\phi = -\{(c - H^2)\phi - Hd\}\overline{g} \tag{6.9}$$

and

$$\nabla^2_{\Sigma}\overline{g}(Z,N) = -\{(c-H^2)\overline{g}(Z,N) + d\}\overline{g},$$

where $d = H\phi - c\overline{g}(Z,N)$, a constant. By Theorem 2 in [Ta], these two equations together with the umbilicity of S yield that either S is a sphere with constant curvature $c - H^2 > 0$ or ϕ and $\overline{g}(Z,N)$ are constant; in the last case, (6.8) implies that $c = H = 0$.

We shall analyse the two cases separately. In the first case, let R_S and R denote the curvature operator of S and $\overline{M}^{n+1}$, respectively. Since, for $u,v \in T\Omega$,

$$R(u,v)Z = -\frac{Z\phi}{\overline{g}(Z,Z)}\{\overline{g}(Z,v)u - \overline{g}(Z,u)v\},$$

by (6.7) we have

$$R(u,v)Z = c\{\overline{g}(v,Z)u - \overline{g}(u,Z)v\} = cR_0(u,v)Z, \tag{6.10}$$

where R_o is defined by the last equality and represents the curvature operator of a spacetime of constant curvature 1. The curvature operator R_S of S satisfies $R_S = (c-H)R_o$.Then, from the Gauss equation and the fact that S is spacelike and umbilical, we get

$$R(u,v)w = R_S(u,v)w + H^2R_0(u,v)w = cR_0(u,v)w, \tag{6.11}$$

for $u,v,w \in TS$. Since Z is transverse to S, except maybe at two points, (6.10) and (6.11) yields

$$R = cR_0,$$

at every point of S. Now, using the Gauss equation for the leaves of the foliation $\mathcal{F}(Z)$, we have

$$\mathcal{R}(u,v)w = (c - \mathcal{H}^2)R_0(u,v)w,$$

for $u,v,w \in T_p\mathcal{F}(Z)$ and $p \in S$, where $\mathcal{R}$ and $\mathcal{H}$ are the curvature operator and the mean curvature of the leaves, respectively.

This implies that

$$\mathcal{R} = (c - \mathcal{H}^2)R_0$$

on Ω, because $\mathcal{R}$ is invariant under the flow of Z_0. By using the Gauss equation once more we obtain

$$R(u, v)w = cR_0(u, v)w,$$

for $u, v, w \in T_x\mathcal{F}(Z), x \in \Omega$. Since Z is orthogonal to the leaves of $\mathcal{F}(Z)$, this equation together with (6.10) imply that

$$R = cR_0, \text{ on } \Omega.$$

That is, the open set $\Omega \subset \overline{M}^{n+1}$ has constant curvature $c > H^2 > 0$ and so it is, locally, isometric to a de Sitter spacetime. Since S is a round sphere umbilically immersed in $\overline{M}^{n+1}$, one concludes that S is a round hypersphere described in Example 3.6 in Section 3, proving that (ii) holds in this case.

It remains to consider the case when $c = H = 0$ and ϕ and $\overline{g}(Z, N)$ are constant on S. Clearly, S is totally geodesic and ϕ is constant on Ω. Using the first Minkowski formula one concludes that $\phi = 0$ and so, Z is parallel on Ω, the foliation $\mathcal{F}(Z)$ has totally geodesic leaves and the flow of Z_0 is isometric. This implies that there exists another parallel vector field and we then prove that $\overline{M}^{n+1}$, locally near to S, is the product $R_1^2 \times Q^{n-1}$ of a Lorentz plane and a Riemannian (n-1)-manifold. Since also Z^T is a parallel field tangent to S, we have that S is, locally, a Riemannian product $R \times Q$, where R is a spacelike line in the Lorentz plane R_1^2, giving the conclusion in (iii) and finishing the proof of the theorem.

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