ON SEMILINEAR PARABOLIC PROBLEMS WITH NON-LIPSCHITZ NONLINEARITIES

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Abstract

We consider a semilinear reaction-diffusion equation with nonlinear terms of non-Lipschitz type. We state and prove some comparison principles for this problem when its domain of definition Ω is bounded. We apply those results to discuss uniqueness and nonuniqueness of solutions. These comparison arguments are also used to obtain analogous results when $\Omega = \mathbb{R}^N$.

Resumo

Consideramos a equação semilinear de reação-difusão com termos não-lineares de tipo não-Lipschitz. Estabelecemos e demonstramos alguns princípios de comparação para este problema quando seu domínio de definição Ω é limitado. Aplicamos estes resultados para discutir a unicidade e não-unicidade de soluções. Estes argumentos de comparação são também utilizados para obter resultados análogos quando $\Omega = \mathbb{R}^N$.

1. Introduction

In this paper we discuss the following semilinear parabolic equation

$$\begin{cases} u_t - \Delta u = g(u) \qquad (t,x) \in (0,T) \times \Omega, \\ u = 0 \qquad (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x) \qquad x \in \Omega. \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ has smooth boundary $\partial\Omega$. More technical details will be given later on. Problem (1.1) has been extensively investigated since the early seventies. The study of the existence of solutions, their regularity and asymptotic

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behavior have called the attention of a large list of researchers. We cite [3], [4], [9], [10] with by no means any intention of covering the subject. One of the main features of parabolic problems is that they preserve some ordering and this a key property to study (1.1). For example, assume that $g : \mathbb{R} \to \mathbb{R}$ verifies the Lipschitz condition $|g(u) - g(v)| \leq k|u - v|$ for all $u, v \in \mathbb{R}$ and that $\overline{u}$ is a corresponding supersolution, i.e., $\overline{u}$ satisfies

$$\begin{cases} \overline{u}_t - \Delta \overline{u} \geq g(\overline{u}) \qquad (t,x) \in (0,T) \times \Omega, \\ \overline{u} \geq 0 \qquad (t,x) \in (0,T) \times \partial\Omega, \\ \overline{u}(0,x) \geq u_0(x) \qquad x \in \Omega. \end{cases} \tag{1.2}$$

(Subsolutions are defined analogously, with reversing signs.) To prove a maximum principle in this situation, we note $w = (u - \overline{u})^+ = \max(u - \overline{u}, 0)$ and, assuming that u and $\overline{u}$ are regular, we subtract (1.1) from (1.2), multiply both sides by w and integrate over Ω to obtain

$$\frac{d}{dt} \int w^2 + \int |\nabla w|^2 \leq \int (g(u) - g(\overline{u}))w \leq k \int w^2.$$

Since $w(0,x) \equiv 0$, we can then apply Gronwall's Lemma to conclude that $w \equiv 0$, that is, $u \leq \overline{u}$. We remark that it suffices to assume that g is locally Lipschitz for the argument to work. On the other hand, this condition is essential (if $\Omega = \mathbb{R}^N$, then any solution of the ODE $u' = g(u)$ also satisfies (1.1), and it is well known that there is no comparison principles for such a problem when g is not locally Lipschitz).

The purpose of this work is to discuss some comparison principles when g is not necessarily Lipschitz. Non-Lipschitz semilinear problems were first studied considering $g(u) = u^q$, $q < 1$, in [8], for $\Omega = \mathbb{R}^N$, and in [7], for bounded domains, both works dealing with uniqueness and non-uniqueness results. Their arguments, however, rely strongly in the precise form of g and it is not clear how to extend them to even mild perturbations of u^q. Extensions of the results of [7] more general functions have been considered in [5]. They are consequence of some comparison principles that hold in the non-Lipschitz case. One interesting example is the function $g(u) = \lambda(u^q + u^p)$, where $0 < q < 1 < p$ and $\lambda >$

0. The stationnary (elliptic) problem associated to (1.1) has been treated in some works, see e.g. [1], [2]. The convergence of the trajectories of (1.1) to these stationnary points is investigated in [5]. We also remark that comparison principles for non-Lipschitz systems of parabolic equations, and some of their consequences, can be found in [6].

In this work we describe some maximum principles presented in [5], some of their applications, and we also prove new results. This in done in section 2. We consider the whole space case $\Omega = \mathbb{R}^N$ in section 3, showing that the uniqueness and non-uniqueness results of [8] can also be extended to more general functions.

2. Bounded domains.

In this section we describe, and extend, the results obtained in [5] for positive solutions of the problem (1.1) in bounded and smooth domains $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$. We assume that

$$g : [0, \infty) \to [0, \infty) \quad \text{is continuous.} \tag{2.1}$$

Given $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, we call a solution of (1.1) a function $u \in L^\infty((0,T)\times \Omega)$ for some $T > 0$, $u \geq 0$ which satisfies

$$u(t) = \mathcal{T}(t)u_0 + \int_0^t \mathcal{T}(t-s)g(u(s))\,ds, \tag{2.2}$$

for all $t \in [0,T]$, where $\mathcal{T}$ is the linear heat semigroup. Note that the above definition makes sense. Indeed, if $u \in L^\infty((0,T)\times\Omega)$, then $g(u) \in L^\infty((0,T)\times \Omega)$, so that the right-hand side of (2.2) is well-defined. Moreover, since $g(u) \in L^\infty((0,T)\times\Omega)$, standard regularity results imply that $u \in C([0,T], L^r(\Omega))$, that $u - \mathcal{T}(t)u_0 \in L^r((0,T), W^{2,r}(\Omega)\cap W_0^{1,r}(\Omega)) \cap W^{1,r}((0,T), L^r(\Omega))$ for every $r < \infty$ and that u satisfies the first equation of (1.1) for a.a. $t \in (0,T)$.

Before discussing any comparison principles, the following result of [5] ensures that (1.1) has a (local) solution, for any regular domain $\Omega \subset \mathbb{R}^N$, not necessarily bounded.

Proposition 2.1. *Assume (2.2). Given $\Omega \subset \mathbb{R}^N$ a regular domain and $u_0 \in$*

$L^\infty(\Omega)$, $u_0 \geq 0$, there exists a larger solution $u \geq 0$ of (1.1) defined for some time interval $[0,T)$, $u \in L^\infty((0,\infty)\times\Omega)$. u is the larger solution in the sense that if $v \geq 0$ is any subsolution of (1.1) on some interval $[0,T]$ and if v is smooth enough (i.e. $v \in L^\infty((0,\infty)\times\Omega)\cap C([0,T],L^2(\Omega))$ and $v \in L^2_{\mathrm{loc}}((0,T),H^1(\Omega))\cap W^{1,2}_{\mathrm{loc}}((0,T),H^{-1}(\Omega))$), then $v(t) \leq u(t)$ for all $t \leq T$.

We limit ourselves to give the idea of the proof of (2.1), which can be found in [5]. Extending g as $g(u) = 0$ for $u \leq 0$, we consider g_M the standard truncation of g. It is easy to see that g_M can be approximate by a nonincreasing sequence of smooth functions $g_{n,M}$. Using the standard theory for each $g_{n,M}$ we produce a nonincreasing sequence of approximate solutions $u_{n,M}$, converging to a solution u_M of (1.1) for $g = g_M$. u_M is in fact the desired solution for M large enough.

We introduce the following condition on g.

$$\begin{aligned}&\text{For all } M > 0, \text{ there exists } L_M < \infty \text{ such that}\\ &g(u) - g(v) \leq \tfrac{L_M}{v}(u-v) \text{for all } 0 < v \leq u \leq M,\end{aligned} \tag{2.3}$$

Note that (2.3) is a one-sided condition, which means when g is C^1 that $g'(u) \leq L_M/u$ for all $u \in (0,M)$. To present our first comparison priciple we define $d_\Omega(x) = \inf\{||x-y||, y \in \partial\Omega\}$ the distance to the boundary of Ω function.

Proposition 2.2. *Assume (2.1) and (2.3) and let $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$. Suppose u is a supersolution of (1.1) and v is a subsolution of (1.1) on some interval $[0,T]$. If u and v are sufficiently smooth, i.e. $u,v \in L^\infty((0,T)\times\Omega)\cap C([0,T],L^2(\Omega))$ and $u,v \in L^2_{\mathrm{loc}}((0,T),H^1(\Omega))\cap W^{1,2}_{\mathrm{loc}}((0,T),H^{-1}(\Omega))$ and if $u(0) \geq \delta d_\Omega$ for some $\delta > 0$, then $u(t) \geq v(t)$ for all $t \in [0,T]$.*

The proof is based on Hardy's inequality

$$\int_\Omega \frac{\varphi^2}{d_\Omega^2} \leq C\int_\Omega |\nabla\varphi|^2, \tag{2.4}$$

for all $\varphi \in H^1_0(\Omega)$. For more details, see [5], where it is also proved the following corollary of Proposition 2.2, treating the case $u_0 = 0$.

Corollary 2.3. *Assume (2.1) and (2.3). Let $u_0 = 0$ and $T > 0$. Let $u,v \in$*

$L^\infty((0,T) \times \Omega) \cap C([0,T], L^2(\Omega))$ *with* $u, v \in L^2_{\mathrm{loc}}((0,T), H^1(\Omega)) \cap W^{1,2}_{\mathrm{loc}}((0,T), H^{-1}(\Omega))$. *If* u *is a positive supersolution of (2.1) and if* v *is a subsolution, then* $u(t) \geq v(t)$ *for all* $0 \leq t \leq T$.

Curious enough, a second comparison principle holds for g having derivatives bounded from below. So, instead of (2.3) we consider the following assumption.

$$\begin{aligned} &\text{Given } M > 0, \text{ there exists } K_M < \infty \text{ such that} \\ &g(u) - g(v) \geq K_M(u - v) \text{ for all } 0 < v \leq u \leq M, \end{aligned} \tag{2.5}$$

Proposition 2.4. *Assume (2.1) and (2.5) and let* $u_0 = 0$. *Suppose* u *is a positive supersolution of (2.1) and* v *is a corresponding subsolution on some interval* $[0,T]$. *If* u *and* v *are sufficiently smooth, i.e.* $u, v \in L^\infty((0,T) \times \Omega) \cap C([0,T], L^2(\Omega))$ *and* $u, v \in L^2_{\mathrm{loc}}((0,T), H^1(\Omega)) \cap W^{1,2}_{\mathrm{loc}}((0,T), H^{-1}(\Omega))$ *then* $u(t) \geq v(t)$ *for all* $t \in [0,T]$.

Proof: It clearly suffices to prove the result for any $T' < T$, so we may assume that $||u||_\infty, ||v||_\infty \leq M$, for some $M > 0$. Using (2.5), we replace $-\Delta$ by $L_M = -\Delta + K_M I$ and g by $g_M = g + K_M$, so that g_M is a nondecreasing function. Let $s > 0$ and consider $\tilde{u}(t) = u(t+s)$. It follows from the standard maximum principle that $\tilde{u}(0) > 0$ in Ω, with external normal derivative $\dfrac{\partial}{\partial \eta}\tilde{u} < 0$ at $\partial\Omega$. Consider $T_* = \sup\{\tau \in [0,T'], \tilde{u}(t) \geq v(t), \text{ for all } t \in [0,\tau]\}$. We claim that $T_* = T'$. Indeed, since $\tilde{u}(0) = u(s)$, we have that $T_* > 0$. Moreover,

$$(\tilde{u} - v)_t + L_M(\tilde{u} - v) \geq g_M(\tilde{u}) - g_M(v) \geq 0.$$

Therefore, from the usual maximum principle we obtain that $\tilde{u} > v$ in Ω, with $\dfrac{\partial}{\partial \eta}\tilde{u} < \dfrac{\partial}{\partial \eta} v$ at $\partial\Omega$ for $t \leq T_*$. Hence, we clearly can not have that $T_* < T'$. Therefore, $\tilde{u} > v$ for all t and we let $s \to 0$ to finish the proof.

□

We now show that the comparison principles (2.3) and (2.4) are efficient tools to analyze the number of solutions of (1.1). Having in mind the model function u^q, we also suppose that

$$g \text{ is concave on } (0,a) \text{ for some } a > 0 \text{ and } \frac{d^+ g}{dx}(0) = +\infty. \tag{2.6}$$

Theorem 2.5. *Suppose g satisfies (2.1), (2.3) and (2.6). Then,*

(i) *if*

$$\int_0^b \frac{ds}{g(s)} < \infty, \tag{2.7}$$

for some $b > 0$, then for all $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, there exists a unique, positive solution u of (1.1) defined on a maximal time interval $[0, T_m)$, $u \in L^\infty((0,T) \times \Omega)$ for all $T < T_m$;

(ii) *if*

$$\int_0^b \frac{ds}{g(s)} = \infty, \tag{2.8}$$

for any $b > 0$ then $u \equiv 0$ is the only solution of (1.1) when $u_0 = 0$. If $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$ and $u_0 \not\equiv 0$, then there exists a unique, positive solution u of (1.1) defined on a maximal time interval $[0, T_m)$, $u \in L^\infty((0,T) \times \Omega)$ for all $T < T_m$.

Proof: (i) was shown in [5] so we prove here only (ii). The existence of a solution has already been discussed and the existence of a maximal time interval $[0, T_m)$ comes from the standard semilinear theory, see e.g. [9]. We need only show the uniqueness result. Consider first $u(0) = 0$ and let $z(t)$ satisfy the ODE $z' = g(z)$, with $z(0) = \delta > 0$. Then $z(t)$ is a positive supersolution of (1.1) and we can use Proposition 2.2 to write that $u \leq z$. It follows from (2.8) that $z(t) \to 0$ as $\delta \to 0$, for all $t > 0$, showing that $u = 0$. Observe that the above argument shows that

$$\text{there is no positive subsolution } w \text{ with } w(0) = 0. \tag{$*$}$$

Let now u and v be two solutions of (1.1) such that $u(0) = v(0)$. We claim that $u = v$ in some small time interval $[0, T)$ and this clearly implies that uniqueness holds in $[0, T_m)$. To show the claim, we remark that (2.3) and (2.6) imply (3.2) below, that is, given $M > 0$ there exists $\varepsilon > 0$ such that $g(u) - g(v) \leq g(u - v)$ for all $0 \leq v \leq u \leq M$ verifying $u - v \leq \varepsilon$ (this is shown in the proof of Theorem 3.1, Step 4). Using Propositon 2.1, we can assume that $w = u - v \geq 0$. Take T small so that $||u||_\infty \leq M$, with $||w||_\infty \leq \varepsilon$. Therefore, $w_t - \Delta w \leq g(w)$, and the result follows from (*) above.

□

Remark 2.6. Under the hypothesis of Theorem (2.5) (i), (1.1) has infinitely many solutions leaving $u_0 = 0$. In fact, we can describe all such solutions in the following way. Let u be the unique positive solution of the problem, $s > 0$ and define $u_s(t)$ as $u_s(t) = u(t-s)$, if $t \geq s$, $u_s(t) = 0$ for $0 \leq t < s$. Then v solves (1.1) with $v(0) = 0$ if and only if $v = u_s$ for some s.

We refer to [5] for some applications of Proposition 2.2 to the study of the asymptotic behavior of the solutions of (1.1).

3. The whole space case

Let again g satisfy (2.1) and $u_0 \in L^\infty(\mathbb{R}^N)$, $u_0 \geq 0$. In this section we consider the problem

$$\begin{cases} u_t - \Delta u = g(u) & (t,x) \in (0,T) \times \mathbb{R}^N, \\ u(0,x) = u_0(x) & x \in \Omega. \end{cases} \tag{3.1}$$

Given $T > 0$, we say that $u \in L^\infty((0,T) \times \Omega)$ is a solution of (3.1) if

$$u(t) = \mathcal{T}(t)u_0 + \int_0^t \mathcal{T}(t-s)g(u(s))\,ds, \tag{3.2}$$

for all $t \in [0,T]$, where $\mathcal{T}$ is the linear heat semigroup. Standard regularity results imply that $u \in C([0,T], L^r_{loc}(\mathbb{R}^N))$, that $u - \mathcal{T}(t)u_0 \in L^r((0,T), W^{2,r}_{loc}(\mathbb{R}^N)) \cap W^{1,r}((0,T), L^r_{loc}(\mathbb{R}^N))$ for every $r < \infty$ and that u satisfies the equation (3.1) for a.a. $t \in (0,T)$.

We will show how comparison arguments can be used in relation to (3.1). Proposition 2.2, as stated, does not make sense in this context and we don't know how to obtain a corresponding result. However, we can use it to show the following equivalent of Theorem 2.5.

Theorem 3.1. *Suppose g satisfies (2.1), (2.3) and (2.6). Then,*

(i) *if* $\displaystyle\int_0^b \frac{ds}{g(s)} < \infty$, *for some* $b > 0$, *then for all* $u_0 \in L^\infty(\mathbb{R}^N)$, $u_0 \geq 0$, *there exists a unique, positive solution* u *of (3.1) defined on a maximal time interval* $[0, T_m)$, $u \in L^\infty((0,T) \times \Omega)$ *for all* $T < T_m$;

(ii) *if* $\displaystyle\int_0^b \frac{ds}{g(s)} = \infty$ *for any* $b > 0$ *then* $u \equiv 0$ *is the only solution of (3.1) for* $u_0 = 0$. *If* $u_0 \in L^\infty(\mathbb{R}^N)$, $u_0 \geq 0$ *and* $u_0 \not\equiv 0$, *then there exists a unique, positive solution* u *of (3.1) defined on a maximal time interval* $[0, T_m)$, $u \in L^\infty((0,T) \times \Omega)$ *for all* $T < T_m$.

Proof: We recall that the existence of a maximal interval $[0, T_m]$ is given by the standard theory [9]. The rest of the proof will be given by steps.

Step 1. We claim that if u is a positive supersolution of (3.1) then for all $s < T_m$ there exists $\delta_s > 0$ such that $u(x, s) \geq \delta_s$ for all $x \in \mathbb{R}^N$. To show this, let $x_0 \in \mathbb{R}^N$, $\Omega' = B(x_0, 1)$, the unitary ball centered at x_0, and $\tilde{u}(t, x) = u(t+s, x)$. Hence, $\tilde{u}$ is a positive supersolution of (1.1) in Ω' and Proposition 2.2 implies that $u(t + s, x_0) \geq \psi(t, 0)$, where ψ is the positive solution of (1.1) in $B(0, 1)$ verifying $\psi(0) = 0$. We let $s \to 0$ to obtain Step 1.

Step 2. We prove that (i) holds for $u_0 = 0$. In fact, since $\displaystyle\int_0^b \frac{ds}{g(s)} < \infty$, there exists a unique $\varphi(t)$, solution of $\varphi' = g(\varphi)$ with $\varphi(0) = 0$, such that $\varphi(t) > 0$ for $t > 0$. We will show that the only positive solution of (3.1) is $\varphi(t)$. Indeed, suppose u is another positive solution and set $\tilde{u}(t, x) = u(t+s, x)$. Let $\tilde{u}$ and φ to be defined over $[0, T']$ and take $T_* = \sup\{\tau \in [0, T'], \tilde{u}(t) \geq \varphi(t),\ \text{for all}\ t \in [0, \tau]\}$. It follows from Step 1 that $T_* > 0$. We can argue as in the proof of Proposition 2.4 to show that $T_* = T'$. Therefore, $\tilde{u}(t) \geq \varphi(t)$ in $[0, T']$. Since we can reverse the roles of u and φ, this gives the uniqueness result.

Step 3. A remark. We observe that, since g satisfies (2.1), (2.3) and (2.6), the following two facts hold. Given $M > 0$, there exists $0 < \varepsilon \leq a$ such that

$$g(u) - g(v) \leq g(u - v), \tag{3.3}$$

for all $0 \le v \le u \le M$ with $u - v \le \varepsilon$ and

$$g(w) - g(u) + g(v) \ge g(w - u + v) \tag{3.3}$$

for all $0 \le v \le u \le M$, $0 \le w \le u$ with $u - v \le w \le \varepsilon$. Note that (3.3) holds when g is concave. Hence (2.6) gives (3.3) for $0 \le v \le u \le a$, $0 \le w \le u$ with $u - v \le w$. If $a < u \le M$, $\dfrac{g(u) - g(v)}{u - v}$ is bounded above while $\dfrac{g(w) - g(w - u + v)}{u - v} \to +\infty$ as $w \to 0$. Therefore, we can choose ε small enough so that (3.3) holds. (3.2) follows from (3.3) by taking $w = u - v$.

Step 4. The uniqueness for (i) in the general case. Let u, v be two solutions of (3.1) and set $w = u - v$. Following Proposition 2.1, we can suppose that $w \ge 0$. Set $M = ||u||_\infty$ and take ε as in Step 3. w verifies $w_t - \Delta w = g(u) - g(v)$, with $w(0) = 0$. Taking T' small enough, we can suppose that $||w(t)||_\infty \le \varepsilon$ for all $t \le T'$, so that (3.3) yields $w_t - \Delta w \le g(w)$. Therefore, we can argue as in Step 2 to get that $w \le \varphi$. Take now $z = \varphi - w$ and use (3.3) to obtain that $z_t - \Delta z = g(\varphi) - g(u) + g(v) \ge g(z)$. Once again, the arguments of Step 2 give that $z = \varphi - w \ge \varphi$, and so $w = 0$ in $[0, T']$. Since we can iterate the argument, $w = 0$ in $[0, T_m]$.

Step 5. The uniqueness for (ii) when $u_0 = 0$. Let u solve (3.1) with $u(0) = 0$ and consider φ the solution of $\varphi' = g(\varphi)$ with $\varphi(0) = \delta > 0$. We proceed as in Step 2, taking $T_* = \sup\{\tau \in [0, T'], \underline{u}(t) \le \varphi(t),$ for all $t \in [0, \tau]\}$ and showing that $T_* = T'$. But $\displaystyle\int_0^b \frac{ds}{g(s)} = \infty$ implies that $\varphi \to 0$ as $\delta \to 0$. Thus $u = 0$.

Step 6. The uniqueness for (ii) for any u_0. Consider $v \le u$ two solutions of (3.1) and set $w = u - v$. Since $w_0 = 0$ given ε as in Step 3, there exists $T > 0$ such that $||w(t)||_\infty < \varepsilon$ for $t < T$. Therefore, (3.2) yields $w_t - \Delta w = g(u) - g(v) \le g(w)$. The result follows from Step 5 for $t < T$ and for all t by iteration.

Steps 1 through 6 give the desired result.

□

Remark 3.2. It is easy to check from the proof of Theorem 3.1 that a corresponding version of Proposition 2.4 holds for $\Omega = \mathbb{R}^N$.

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