A FAMILY OF JUST-NONSOLVABLE TORSION-FREE GROUPS DEFINED ON n-ARY TREES

Said Sidki* **Edméia F. da Silva**

Abstract

We generalize the construction of a just-nonsolvable torsion-free group defined on the binary tree given by A.M. Brunner, Said Sidki and Ana Cristina Vieira, producing an infinite family of two generated groups of this type acting on n-ary trees, whose generators are conjugates of the *n-ary adding machine.* These groups have the properties of being torsion-free, nonsolvable, just-nonsolvable and residually "torsion-free, abelian-by-finite and solvable".

Resumo

Neste artigo generalizamos a construção de um grupo livre-de-torção e quase não-solúvel dada por A. M. Brunner, Said Sidki e Ana Cristina Vieira, produzindo uma família infinita de grupos 2-gerados desse tipo agindo sobre em árvores n-árias, cujos geradores são conjugados da *máquina de adição n-ária.* Estes grupos possuem as propriedades de serem livres-de-torção, quase não-solúveis e residualmente " livre-de-torção, abeliano-por-finito e solúvel".

1. Introduction

Certain Burnside groups, such as the 2-group of Grigorchuk and p-groups of Gupta-Sidki, [4] and [5], are just-infinite and nonsolvable. Therefore, these are also examples of torsion groups which are just- nonsolvable, indeed, just non-nilpotent. We remind the reader that given a group property P, a group G is said to be "just non-P" provided the group itself does not satisfy P, yet every proper quotient of it satisfies P. In [3], Brunner-Sidki-Vieira constructed

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a group $\mathcal{H}(2)$ which is 2-generated, torsion-free and just-nonsolvable, as a group of finite state automorphisms of the infinite binary tree. As this group is the first of its kind, we deemed it worthwhile to produce an infinite family of groups with such properties, which was the main theme of the thesis [9].

Following the construction in [3] for the binary tree, let σ be the cyclic permutation $(1, 2, \cdots, n)$ extended rigidly to an automorphism of the n-ary tree. Define the automorphisms of the n-ary tree

$$\rho = (e, \cdots, e, \rho)\sigma, \mu = (e, \cdots, e, \mu^{-1})\sigma.$$

Then, ρ is the n-ary adding machine and μ is a conjugate of ρ. Let $\mathcal{H}(n)$ be the group generated by ρ and μ. The group constructed in [3] is $\mathcal{H}(2)$ and to a certain degree it is canonical. For as it has been shown in [6] that any infinite cyclic group of automorphisms of the binary tree that is finite-state and state-closed is conjugate to one generated by an n-th root of τ or μ, for some odd number n.

The development of the material here follows closely that of [3], however, often the results require special modifications and their proofs delicate readjustments. The first level stabilizer of the n-ary tree in $\mathcal{A}(n)$ is $\mathcal{A}_1(n) = \underbrace{\mathcal{A}(n) \times \cdots \times \mathcal{A}(n)}_{n} = \underset{n}{\times}\mathcal{A}(n)$. This explains the notation $\underset{n}{\times}\mathcal{K}$ for ny subgroup $\mathcal{K}$ of $\mathcal{A}(n)$, which appears in the sequel.

Theorem A. (i) *Each $\mathcal{H}(n)$ is torsion-free and just-nonsolvable.*

(ii) *The second commutator subgroup $\gamma_2(\mathcal{H}(n))$ of $\mathcal{H}(n)$ decomposes as a semidirect product*

$$\gamma_2(\mathcal{H}(n)) = (\underset{n}{\times}\gamma_2(\mathcal{H}(n))) \left\langle c, c^{\rho}, \cdots, c^{\rho^{n-2}} \right\rangle,$$

where $c = [\mu^{-1}, \rho]$ and $\left\langle c, c^{\rho}, \cdots, c^{\rho^{n-2}} \right\rangle$ is a free abelian group of rank $n-1$;

(iii) *The third commutator subgroup of $\gamma_3(\mathcal{H}(n))$ contains the normal subgroup $\underset{n}{\times}\gamma_3(\mathcal{H}(n))$ and the quotient group $\dfrac{\gamma_3(\mathcal{H}(\mathrm{n}))}{\underset{n}{\times}\gamma_3(\mathcal{H}(n))}$ is abelian of rank $2n-2$;*

(iv) *The quotient group* $\dfrac{\mathcal{H}(n)}{\gamma_3(\mathcal{H}(n))}$ *is isomorphic to the 2-generated free nilpotent group of class 2;*

(v) *The central quotient* $\dfrac{\gamma_i(\mathcal{H}(n))}{\gamma_{i+1}(\mathcal{H}(n))}$ *has exponent a divisor of* $2n^2$ *for all* $i \geq 3$.

Furthermore, we prove in this paper

Theorem B. (i) *The group* $\mathcal{H}(n)$ *is residually "torsion-free, abelian-by-finite and solvable" and is nonsolvable.*

(ii) *The subgroup* $\gamma_3(\mathcal{H}(n)) = O^{\{2n\}'}(\mathcal{H}(n))$, *where*

$$O^{\{2n\}'}(\mathcal{H}(n)) = \bigcap \{K \trianglelefteq \mathcal{H}(n) \mid \mathcal{H}(n)/K \ \text{ has order coprime with } \ 2n\}.$$

Similar to $\mathcal{H}(2)$, the group $\mathcal{H}(n)$, $n \geq 3$, admits a *finite endomorphic presentation.* This means that it affords a presentation which is given by a finite generating set, say of size m, a finite set of initial relations together with their images under the iterated action of a finite number of endomorphisms of the free group of rank m (see [1]) .

Theorem C. *The group* $\mathcal{H}(n) =< \rho, \lambda >$ *admits the endomorphic presentation:*

$$\mathcal{H}(n) =< \rho, \lambda \mid \ \varepsilon^k([\lambda, \lambda^{\rho^i}], 1 \leq i \leq [3n/2], i \neq n, k \geq 0 >,$$

where ε *is a substitution defined by:*

$$\begin{array}{rll} \varepsilon : & \rho & \mapsto \rho^n \\ & \lambda & \mapsto \rho^n \lambda^{-1} \rho^n. \end{array}$$

The proof of this theorem can be found in [9]. Another result which can be found in [9] is that the Brunner-Sidki-Vieira group has restricted subgroup structure and, in particular, it does not contain free noncyclic subgroups. The same questions have still to be considered for the group $\mathcal{H}(n)$ where $n \geq 3$.

A problem of more general nature which is suggested by these constructions concerns the existence of finitely generated groups of automorphims of some

n-ary tree which satisfy the properties of being finitely generated, torsion-free, and just-nonnilpotent. It is worth noting that by Theorem 15.3.6 in [8] such groups cannot be solvable.

2. Automorphisms of Regular Trees

We follow the notation and terminology of [8]. The one-rooted regular tree $\mathcal{T}_n$ is the graph of the metric space $(\mathcal{M}, \leq)$, where $\mathcal{M}$ is the free monoid, freely generated by $\mathcal{Y} = \{0, 1, 2, \cdots, n-1\}$, with neutral element ϕ, the empty set, and ordered by the relations: $u \leq v$ if and only if v is a prefix of u. The length of an element $u \in \mathcal{M}$ is indicated by $|u|$. For each $u \in \mathcal{M}$ the set $u\mathcal{M} = \{uv | v \in \mathcal{M}\}$ forms a subtree isomorphic to the original tree $\mathcal{T}_n$.

An automorphism α of $\mathcal{T}_n$ is a bijection on the vertices, wich preserves the order relation. We denote by $\mathcal{A}(n) = Aut(\mathcal{T}_n)$ the automorphism group of $\mathcal{T}_n$ and by $P(\mathcal{Y})$ the group of permutations of the set $\mathcal{Y}$.

An automorphism that fixes the first level of vertices of the tree is represented by $(\alpha_0, \alpha_1, \cdots, \alpha_{n-1})$, where α_i is an automorphism of the subtree $i\mathcal{M}$, $0 \leq i \leq n-1$. As such, a subtree is isomorphic to $\mathcal{T}_n$; we identify the α_i's with automorphisms of $\mathcal{T}_n$. An automorphism α of the tree can be represented by $(\alpha_0, \alpha_1, \cdots, \alpha_{n-1})\sigma$, where $\sigma \in P(\mathcal{Y})$ is extended rigidly to an automorphism of $\mathcal{T}_n$ by in the following sense (we write the action on the left):

$$u = yu' \overset{\sigma}{\mapsto} u^\sigma = y^\sigma u', \; \forall u = yu', \text{ with } y \in \mathcal{Y}, u \in \mathcal{M}.$$

Similarly, each α_i has such representation. Successive developments of α produce, for every $u \in \mathcal{M}$, an automorphism $\alpha_u = \alpha_u(\alpha)$ of the tree, and a permutation $\sigma_u = \sigma_u(\alpha)$ of $\mathcal{Y}$. The set $\mathcal{Q}(\alpha) = \{\alpha_u | u \in \mathcal{M}\}$ is called the set of states of α. We say that a state α_u of α is active if $\sigma_u(\alpha) \neq e$. Otherwise, we say that α_u is inactive.

Given $u = yu' \in \mathcal{M}$ and $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_{n-1})\sigma$ an automorphism of $\mathcal{T}_n$, the image of u under the action of α is given by :

$$u = yu' \overset{\alpha}{\mapsto} u^\alpha = y^\sigma (u')^{\alpha_y}.$$

Let σ be the permutation $(0, 1, 2. \cdots, n-1)$ and define the automorphism of the n-ary tree $\rho = (e, \cdots, e, \rho)\sigma$. Then ρ corresponds to the "n-ary adding machine", since its action on $\mathcal{T}_n$ is given by:for any $u = yu' \in \mathcal{M}$,

$$\rho : u = yu' \mapsto \begin{cases} (y+1)u' & \text{if} \quad 0 \leq y \leq n-2 \\ 0(u')^{\rho} & \text{if} \ \ y = n-1. \end{cases}$$

Since $\rho^n = (\rho, \cdots, \rho)$, the n-ary adding machine is transitive on all the levels of $\mathcal{T}_n$. Also, it can be shown that any automorphism which is transitive on all levels is conjugate to ρ.

Given $\sigma \in P(\mathcal{Y})$, define inductively the following automorphism of $\mathcal{T}_n$: $\sigma_0 = \sigma, \quad \sigma_1 = (\sigma, e, \cdots, e)$ and $\sigma_{i+1} = (\sigma_i, e, \cdots, e), \quad i \geq 1$. We observe that the subgroup $< \sigma_i | 0 \leq i \leq m >$ is isomorphic to $\wr_k C_2$, the wreath product of the cyclic group C_2 of order 2 iterated m times.

3. About the group $\mathcal{H}(n)$

Let $\mathcal{H}(n)$ be the subgroup of $\mathcal{A}(n)$, generated by $\rho = (e, \cdots, e, \rho)\sigma$ and by $\mu = (e, \cdots, e, \mu^{-1})\sigma$.

Since ρ and μ are active, it is convenient to replace μ by the inactive element $\lambda = \rho\mu^{-1}$. We calculate

$$\begin{aligned} \lambda &= (e, \cdots, e, \rho)\sigma \cdot \sigma^{-1}(e, \cdots, e, \mu) \\ &= (e, \cdots, e, \rho\mu) \\ &= (e, \cdots, e, \rho\lambda^{-1}\rho)\,. \end{aligned}$$

The first level stabilizer subgroup $\mathcal{H}_1(n)$ of $\mathcal{H}(n)$ is generated by $\left\{\rho^n,\ \lambda^{\rho^i}, 0 \leq i \leq 3n-1\right\}$. Therefore,

(i) $\mathcal{H}(n) = \mathcal{H}_1(n) < \rho >, [\mathcal{H}(n) : \mathcal{H}_1(n)] = n,$

(ii) $\mathcal{H}_1(n)$ projects in each coordinate onto $\mathcal{H}(n)$, that is, $\mathcal{H}_1(n)$ it is a subdirect product of $\times_n \mathcal{H}(n) = \underbrace{\mathcal{H}(n) \times \cdots \times \mathcal{H}(n)}_{n}$.

(iii) $\mathcal{H}(n)$ is a subgroup of $(\underset{n}{\times}\mathcal{H}(n)) < \sigma >$, and more generally of

$$\underset{n^k}{\times}\mathcal{H}(n) < \sigma_i,\ 0 \leq i \leq k-1 >, \quad \forall\ k \geq 1.$$

Let Λ be the normal closure of λ in $\mathcal{H}(n)$ and let λ_i denote the conjugate λ^{ρ^i} of λ by ρ^i for all $i \in \mathbb{Z}$. It is easy to check that

$$\lambda^{\rho^i} = (e, \cdots, e, \underbrace{\lambda^{-1}\rho^2}_{i-1}, e, \cdots, e),\ 1 \leq i \leq n,$$

and that $[\lambda_i, \lambda_j] = e$, for $i \not\equiv j \ mod\ n$.

Define $\Lambda_i = \langle \lambda_m \,|m \equiv\ i\, mod\ n\rangle$. It follows from the last observation that:

$$\begin{aligned} \Lambda_i \ \text{ commutes with } \Lambda_j, \text{ for } 0 \ &\leq \ i < j \leq n-1; \\ \Lambda \ &= \ \Lambda_0 + \cdots + \Lambda_{n-1}. \end{aligned}$$

Proposition 1. *The group Λ is $3n$-generated.*

Proof. We note that

$$\begin{aligned} \rho^{-n}\lambda\rho^{-n} &= (\rho^{-2}, \cdots, \rho^{-2}, \lambda^{-1}), \\ (\rho^{-n}\lambda\rho^{-n})^{\rho^n} &= \left(\rho^{-2}, \cdots, \rho^{-2}, \lambda_1^{-1}\right). \end{aligned}$$

Therefore, $\left[\rho^{-n}\lambda\rho^{-n}, (\rho^{-n}\lambda\rho^{-n})^{\rho^n}\right] = \left(e, \cdots, e, \left[\lambda^{-1}, \lambda_1^{-1}\right]\right) = e.$

On the other hand,

$$\begin{aligned} \left[\rho^{-n}\lambda\rho^{-n}, (\rho^{-n}\lambda\rho^{-n})^{\rho^n}\right] &= [\rho^{-n}\lambda\rho^{-n}, \rho^{-2n}\lambda] \\ &= \rho^n\lambda^{-1}\rho^n\lambda^{-1}\rho^{2n}\rho^{-n}\lambda\rho^{-n}\rho^{-2n}\lambda \\ &= \lambda_{-n}^{-1}\lambda_{-2n}^{-1}\lambda_{-3n}\lambda_0. \end{aligned}$$

Hence, $\lambda_{-n}^{-1}\lambda_{-2n}^{-1}\lambda_{-3n}\lambda_0 = e.$ On conjugating by ρ^{3n} we obtain

$$\lambda_{3n} = \lambda_0^{-1}\lambda_n\lambda_{2n}.$$

It follows that

$$\Lambda =< \lambda_i \,|0 \leq i \leq 3n-1 > .$$

$\square$

The permutation $\sigma = (0, 1, 2, \cdots, n-1)$ is inverted by

$$\delta = \prod_{i=0}^{\left[\frac{n}{2}\right]-1} (i, n-i-1).$$

Given an automorphism α of the tree $\mathcal{T}_n$, we define $\alpha^{(1)}$ to be the element $(\alpha, \cdots, \alpha)$ of the first level stabilzer and define inductively, $\alpha^{(k+1)} = (\alpha^{(k)})^{(1)}$ for all $k \geq 0$.

Lemma 1. *Let δ be the permutation defined above. Then the automorphism $\theta = \theta^{(2)}\delta^{(1)}$ of the n-ary tree has order 2 and conjugates ρ to μ.*

Proof. First of all we observe that the states of θ are θ and $\theta^{(1)}\delta$. Since the square of θ is $\theta^2 = \left((\theta^2)^{(1)}, (\theta^2)^{(1)}, \ldots, (\theta^2)^{(1)}\right)$ and as its states are inactive, it follows that $\theta^2 = e$.

We have

$$\rho^\theta = (e, \cdots, e, \rho^{\theta^{(1)}\delta})\sigma,$$

and

$$\begin{aligned} \rho^{\theta^{(1)}\delta} &= (\theta, \cdots, \theta)\delta(e, \cdots, e, \rho)\sigma(\theta, \cdots, \theta)\delta \\ &= (\theta, \cdots, \theta)(\rho, e, \cdots, e)\sigma^{-1}(\theta, \cdots, \theta) \\ &= (\rho^\theta, e, \cdots, e)\sigma^{-1}; \end{aligned}$$

therefore,

$$\rho^\theta \rho^{\theta^{(1)}\delta} = \left(e, e, \cdots, e, \rho^{\theta^{(1)}\delta}\rho^\theta\right).$$

As the states of $\rho^\theta \rho^{\theta^{(1)}\delta}$ are inactive, $\rho^\theta \rho^{\theta^{(1)}\delta} = e$, or equivalently, $(\rho^\theta)^{-1} = \rho^{\theta^{(1)}\delta}$. Then $\rho^\theta = (e, \cdots, e, (\rho^\theta)^{-1})\sigma$ and therefore $\rho^\theta = \mu$.

□

4. Decomposition of the subgroups $\gamma_2(\mathcal{H}(n))$ and $\gamma_3\,(\mathcal{H}(n))$

We define for all $n \geq 3$, the commutator $c = [\lambda, \rho] = \lambda^{-1}\rho^{-1}\lambda\rho$ and the sequence of the automorphisms

$$c_1 = (c, e, \cdots, e), \;\; c_{i+1} = (c_i, e, \cdots, e), \;\; \text{for } \; i \geq 1, \;\; (\text{vectors of length } n^{i+1}).$$

Let $\gamma_i(\mathcal{H}(n)),\ i \geq 1$, be the lower central series of $\mathcal{H}(n)$.

Proposition 2. *The subgroup $\gamma_2(\mathcal{H}(n))$ of $\mathcal{H}(n)$ decomposes as a semidirect product*

$$\gamma_2(\mathcal{H}(n)) = (\underset{n}{\times}\gamma_2(\mathcal{H}(n)))\left\langle c, c^{\rho}, \cdots, c^{\rho^{n-2}}\right\rangle,$$

where $\left\langle c, c^{\rho}, \cdots, c^{\rho^{n-2}}\right\rangle$ *is free abelian of rank* $n-1$.

Proof. We have

$$c = [\lambda, \rho] = (\lambda^{-1}\rho^2, e, \cdots, e, \rho^{-1}\lambda\rho^{-1}),$$

and therefore $[c, \lambda] = e$ follows.

We calculate

$$\begin{array}{rcl} [\lambda, \rho^n] & = & (e, \cdots, e, [\rho\lambda^{-1}\rho, \rho]) \\ & = & (e, \cdots, e, (c^{-1})^{\rho}) \\ & = & (e, \cdots, e, c^{-1})^{\rho^n} \end{array}.$$

Therefore, $c_1 = (c, e, \cdots, e) = [\lambda, \rho^n]^{-\rho^{-2n+1}} \in \gamma_2(\mathcal{H}(n))$.

As $\gamma_2(\mathcal{H}(n))$ is the normal closure of c in $\mathcal{H}(n)$ and $\mathcal{H}_1(n)$ projects in each of it coordinates onto $\mathcal{H}(n)$, we see that

$$\gamma_2(\mathcal{H}(n)) \times \{e\} \times \cdots \times \{e\}$$

is the normal closure of c_1 in $\mathcal{H}_1(n)$. Therefore, the normal closure of c_1 in $\mathcal{H}(n)$ is $\underset{n}{\times}\gamma_2(\mathcal{H}(n))$ which is a subgroup of $\gamma_2(\mathcal{H}(n))$.

Since

$$[\lambda, \rho^n] = cc^{\rho} \cdots c^{\rho^{n-1}} \in \underset{n}{\times}\gamma_2(\mathcal{H}(n)),$$

it follows that

$$c^{\rho^{n-1}} \equiv c^{-\rho^{n-2}} \cdots c^{-\rho}c^{-1} \ mod \underset{n}{\times}\gamma_2(\mathcal{H}(n)).$$

Furthermore, since

$$c^{\rho^i} = (e, \cdots, e, \underbrace{\rho^{-2}\lambda}_{i-1}, \lambda^{-1}\rho^2, e, \cdots, e), 0 \leq i \leq n-1,$$

$[\lambda_i, \lambda_j] = e$, for $i \not\equiv j \ mod \ n$ and $[c, \lambda] = e$,
we have

$$\left[c^{\rho^i}, c^{\rho^j}\right] = e, \ 1 \leq i, j \leq n-2.$$

Now the decomposition of $\gamma_2(\mathcal{H}(n))$ follows .

$\square$

Proposition 3. *The third term $\gamma_3(\mathcal{H}(n))$ of the lower central series of $\mathcal{H}(n)$, has the factorization*

$$\gamma_3(\mathcal{H}(n)) = \underset{n}{\times}\gamma_3(\mathcal{H}(n)) \left\langle [c,\rho]^{\rho^i}, [c,\rho^n]^{\rho^i}, 0 \leq i \leq n-2 \right\rangle,$$

where $[c,\rho^n] \equiv (c^{-1}, e, \cdots, c)$ mod $\underset{n}{\times}\gamma_3(\mathcal{H}(n))$ and $\dfrac{\gamma_3(\mathcal{H}(n))}{\underset{n}{\times}\gamma_3(\mathcal{H}(n))}$ is abelian of rank $2n-2$.

Proof. As $c_1 = (c, e, \cdots, e)$ belongs to $\gamma_2(\mathcal{H}(n))$, it follows that $[c_1, \rho^n] = ([c,\rho], e, \cdots, e)$ is in $\gamma_3(\mathcal{H}(n))$.

The normal closure of $[c,\rho]$ in $\mathcal{H}(n)$ is equal to $\gamma_3(\mathcal{H}(n))$, because $[c,\lambda] = e$. Thus, the normal closure of $[c_1,\rho^n]$ in $\mathcal{H}_1(n)$ is $\gamma_3(\mathcal{H}(n)) \times \{e\} \cdots \times \{e\}$. Therefore, $\gamma_3(\mathcal{H}(n)) \times \{e\} \cdots \times \{e\}$ is a subgroup of $\gamma_3(\mathcal{H}(n))$ and $\underset{n}{\times}\gamma_3(\mathcal{H}(n))$ is a normal subgroup of $\mathcal{H}(n)$.

It is easy to see that $[c,\rho]$ and λ commute modulo $\underset{n}{\times}\gamma_3(\mathcal{H}(n))$. Furthermore

$$\begin{aligned} [c,\rho^n] &= ([\lambda^{-1}\rho^2, \rho], e, \cdots, e, [\rho^{-1}\lambda\rho^{-1}, \rho]) \\ &= (c^{-\rho^2}, e, \cdots, e, c^{\rho^{-1}}) \\ &\equiv (c^{-1}, e \cdots, c) \ mod \ \underset{n}{\times}\gamma_3(\mathcal{H}(n)). \end{aligned}$$

From

$$[c,\rho^n] = [c,\rho][c,\rho]^{\rho} \cdots [c,\rho]^{\rho^{n-1}},$$

we have

$$[c,\rho]^{\rho^{n-1}} = [c,\rho]^{-\rho^{n-2}} \cdots [c,\rho]^{-1}[c,\rho^n].$$

To complete the description of $\gamma_3(\mathcal{H}(n))$, we note that

$$[c,\rho]^{\rho^i} = (e, e, \cdots, e, \underbrace{\lambda^{-1}\rho^2}_{i-1}, (\rho^{-2}\lambda)^2, \lambda^{-1}\rho^2, e, \cdots, e),$$

and that

$$[c,\rho^n]^{\rho^i} = (e, \cdots, e, \underbrace{c}_{i-1}, c^{-1}, e, \cdots, e) \ mod \ \underset{n}{\times}\gamma_3(\mathcal{H}(n)),$$

for every $0 \leq i \leq n-2$.

We observe that modulo $\underset{n}{\times}\gamma_3(\mathcal{H}(n))$,

(i) $[c,\rho^n]^{\rho^{n-1}} = (e,\cdots,e,c,c^{-1}) = [c,\rho^n]^{-1}[c,\rho^n]^{-\rho}\cdots[c,\rho^n]^{-\rho^{n-2}}$,

(ii) $[c,\rho]^{\rho^i}$ commutes with $[c,\rho]^{\rho^j}$, where $0 \leq i,\, j \leq n-2$.

As

$$[c,\rho^n,\lambda] = (e,\cdots,e,[c^{\rho^{-1}},\rho\lambda^{-1}\rho])$$

is an element of $\underset{n}{\times}\gamma_3(\mathcal{H}(n))$, the assertion about the decomposition of $\gamma_3(\mathcal{H}(n))$ is now verified.

□

Proposition 4. *The centralizer of $\gamma_2(\mathcal{H}(n))$ in $\mathcal{H}(n)$ is trivial.*

Proof. Let $\omega = \omega(\rho,\lambda)$ be an element of $\mathcal{H}(n)$ which centralizes $\gamma_2(\mathcal{H}(n))$. Then

$$\omega = (\omega_0,\omega_1,\ldots,\omega_{n-1})\sigma^i,\ 0 \leq i \leq n-1,$$

centralizes $c_1 = (c,e,\ldots,e)$, which implies that $i=0$.

Since $\underset{n}{\times}\gamma_2(\mathcal{H}(n))$ is a subgroup of $\mathcal{H}(n)$ we see that each ω_i centralizes $\gamma_2(\mathcal{H}(n))$ and so each ω_i is inactive.

Repeating the argument above with ω_i in the place of ω, we see that each state of ω_i centralizes $\gamma_2(\mathcal{H}(n))$ and therefore is inactive. After a finite number of repetitions, we conclude that all the states of ω are inactive, which implies that ω is trivial.

□

Theorem 1. *The quotient group $\dfrac{\mathcal{H}(n)}{\gamma_3(\mathcal{H}(n))}$ is isomorphic to the 2-generated free nilpotent group of class 2.*

Proof. (i) The quotient group $\dfrac{\mathcal{H}(n)}{\gamma_2(\mathcal{H}(n))}$ is torsion-free of rank 2.

Let i and j be integers such that $u = \lambda^i\rho^j \in \gamma_2(\mathcal{H}(n))$.

Since $\gamma_2(\mathcal{H}(n)) \leq \mathcal{H}_1(n)$, $j = nj_1$ for some $j_1 \in \mathbb{Z}$. Therefore

$$u = (\rho^{j_1}, \cdots, \rho^{j_1}, c'\lambda^{-i}\rho^{2i+j_1})$$

for a certain $c' \in \gamma_2(\mathcal{H}(n))$.

Using Proposition 2, there exist integers $k_0, k_1, \cdots, k_{n-2}$ and automorphisms $c_0, c_1, \cdots, c_{n-1} \in \gamma_2(\mathcal{H}(n))$ such that

$$\begin{aligned} u \;=\; & (c_0\lambda^{-(k_0-k_1)}\rho^{2(k_0-k_1)}, c_1\lambda^{-(k_1-k_2)}\rho^{2(k_1-k_2)}, \cdots, \\ & c_{n-3}\lambda^{-(k_{n-3}-k_{n-2})}\rho^{2(k_{n-3}-k_{n-2})}, c_{n-2}\lambda^{-k_{n-2}}\rho^{2k_{n-2}}, c_{n-1}\lambda^{k_0}\rho^{-2k_0}), \end{aligned}$$

from which it follows, on comparing the two expressions of u that

$$\lambda^{k_0+i}\rho^{-(2k_0+2i+j_1)}, \lambda^{-k_{n-2}}\rho^{2k_{n-2}-j_1}, \lambda^{k_r-k_{r+1}}\rho^{2(k_{r+1}-k_r)-j_1}$$

are elements of $\gamma_2(\mathcal{H}(n))$, with $0 \leq r \leq n-3$. Thus

$$\begin{aligned} & \left(\prod_{r=0}^{n-3} \lambda^{-(k_r-k_{r+1})}\rho^{2(k_r-k_{r+1})-j_1}\right)\lambda^{-k_{n-2}}\rho^{2k_{n-2}-j_1}\lambda^{k_0+i}\rho^{-(2k_0+2i+2j_1)} \\ & \equiv \lambda^i\rho^{-2i-j}\, mod\, \gamma_2(\mathcal{H}(n)), \end{aligned}$$

which implies that $\lambda^i\rho^{-(2i-j)} \in \gamma_2(\mathcal{H}(n))$.

Since $\lambda^i\rho^j \in \gamma_2(\mathcal{H}(n))$, it follows from the previous conclusions that $\rho^{2(i+j)}$ is in $\gamma_2(\mathcal{H}(n))$. Therefore n divides $2(i+j)$, and since n divides j, n divides $2i$.

We conclude then that, given i, j integers such that $\lambda^i\rho^j \in \gamma_2(\mathcal{H}(n))$, n divides both j and $2i$.

Consequentely, n divides $2(k_r - k_{r+1}) - j_1$ and n divides $2(k_r - k_{r+1})$, for all $0 \leq r \leq n-3$. Therefore $n|j_1$ and $n^2|j$. Since $\rho^{2(i+j)} \in \gamma_2(\mathcal{H}(n))$, on replacing $\lambda^i\rho^j$ in the above argument by $\rho^{2(i+j)}$, we see that $n^2|2(i+j)$ and that the same hold for $2i$.

Again, as $\lambda^{-(k_r-k_{r+1})}\rho^{2(k_r-k_{r+1})-j_1} \in \gamma_2(\mathcal{H}(n))$, for all $0 \leq r \leq n-3$, it follows that $n^2|2(k_r - k_{r+1}) - j_1$ and $n^2|2(k_r - k_{r+1})$, and so n^2 divides j_1 and consequently j is multiple of n^3. As before, we see that $n^3|2(i+j)$ and therefore $n^3|2i$.

Repeating this process, we conclude that $n^s|j$ and $n^s|2i$ for all $s \geq 1$. Hence, $i = j = 0$.

(ii) The quotient $\dfrac{\gamma_2(\mathcal{H}(n))}{\gamma_3(\mathcal{H}(n))}$ is an infinite cyclic group.

Of course, $c \notin \gamma_3(\mathcal{H}(n))$, otherwise, $\gamma_2(\mathcal{H}(n)) = \gamma_i(\mathcal{H}(n))$ for every $i \geq 1$. But, by Proposition 2 and Proposition 3, we see that the group $\dfrac{\gamma_2(\mathcal{H}(n))}{\times_n \gamma_2(\mathcal{H}(n))}$ has rank $n-1$ and the group $\dfrac{\gamma_3(\mathcal{H}(n))}{\times_n \gamma_3 \mathcal{H}(n)}$ has rank $2(n-1)$.

We know that

$$[\lambda, \rho^n] = cc^{\rho} \cdots c^{\rho^n} = (e, \cdots, e, c^{-\rho}),$$

is an element of $\gamma_2(\mathcal{H}(n))$, therefore,

$$c^n \equiv (e, \cdots, e, c^{-1}) \, mod \, \gamma_3(\mathcal{H}(n))$$

and

$$c^n \equiv c^{n\rho} \equiv (c^{-1}, e, \cdots, e) \, mod \, \gamma_3(\mathcal{H}(n)).$$

Now, we will show that if $c^k \in \gamma_3(\mathcal{H}(n))$ for some k, then n divides k.

Let k be a nonnegative integer such that $c^k \in \gamma_3(\mathcal{H}(n))$.

From the factorization of $\gamma_3(\mathcal{H}(n))$, there exist integers $k_0, k_1, \cdots, k_{n-2}$ and intergers $l_0, l_1, \cdots, l_{n-2}$ and $c_0, c_1, \cdots, c_{n-1} \in \gamma_3(\mathcal{H}(n))$ such that

$$\begin{array}{rcl}
(\lambda^{-1}\rho^2)^k & = & c_0(\rho^{-2}\lambda)^{2k_0-k_1}c^{-l_0+l_1}, \\
e & = & c_1(\lambda^{-1}\rho^2)^{k_0-2k_1+k_2}c^{-l_1+l_2}, \\
 & \vdots & \\
e & = & c_i(\lambda^{-1}\rho^2)^{k_{i-1}-2k_i+k_{i+1}}c^{-l_i+l_{i+1}}, \ \ 1 \leq i \leq n-3, \\
 & \vdots & \\
e & = & c_{n-2}(\lambda^{-1}\rho^2)^{k_{n-3}-2k_{n-2}}c^{-l_{n-2}}, \\
(\rho^{-1}\lambda\rho^{-1})^k & = & c_{n-1}c^{k_0}(\lambda^{-1}\rho^2)^{k_0+k_{n-2}}c^{l_0}.
\end{array}$$

Since $\dfrac{\mathcal{H}(n)}{\gamma_2(\mathcal{H}(n))}$ is torsion-free, we have the following system with $n-1$

variables and n equations

$$\begin{array}{lcl} 2k_0 - k_1 + k & = & 0, \\ k_0 - 2k_1 + k_2 & = & 0, \\ \vdots & & \\ k_{i-1} - 2k_i + k_{i+1} & = & 0, \\ \vdots & & \\ k_{n-4} - 2k_{n-3} + k_{n-2} & = & 0, \\ k_{n-3} - 2k_{n-2} & = & 0, \\ k_0 + k_{n-2} + k & = & 0. \end{array}$$

Such system admits an integer solution if, and only if, n divides k, since $nk_0 + (n-1)k = 0$, and this follows from the fact that $k_j = (j+1)k_0 + jk$, for all $1 \leq j \leq n-3$.

Therefore, $k = nk'_1$ and

$$c^k = c^{nk'_1} \equiv (c^{-k'_1}, e, \dots, e) \ mod\, \gamma_3(\mathcal{H}(n)).$$

Again, by Proposition 3, there exist integers $l_0, l_1, \dots, l_{n-2}$ such that

$$c^{k_1+l_0-l_1}, c^{l_{n-2}}, c^{l_0}, c^{l_i-l_{i+1}} \in \gamma_3(\mathcal{H}(n)), \text{ for} 1 \leq i \leq n-3,$$

which implies that $n|k'_1$ and therefore $n^2|k$. Repeating this procedure we prove that $n^s|k$ for all $s \geq 1$. Hence, $k = 0$.

□

Theorem 2. *The central quotient* $\dfrac{\gamma_i(\mathcal{H}(n))}{\gamma_{i+1}(\mathcal{H}(n))}$ *has exponent a divisor of* $2n^2$, *for all* $i \geq 3$.

Proof. It is sufficient to prove the assertion for $\dfrac{\gamma_3(\mathcal{H}(n))}{\gamma_4(\mathcal{H}(n))}$.

Since

$$\begin{array}{lcl} \rho^{-n}\lambda\rho^{-n} & = & (\rho^{-2}, \cdots, \rho^{-2}, \lambda^{-1}), \\ \rho^{-2n}\lambda & = & (\rho^{-n}\lambda\rho^{-n})^{\rho^n} \\ & = & (\rho^{-2}, \cdots, \rho^{-2}, \lambda_1^{-1}), \end{array}$$

and $[\lambda, \lambda_1] = e$, it follows that $[\rho^{-n}\lambda\rho^{-n}, \rho^{-2n}\lambda] = e$. Now

$$\begin{array}{lcl} [\rho^{-n}\lambda\rho^{-n}, \rho^{-2n}\lambda] & = & [\rho^{-2n}\lambda[\lambda, \rho^{-n}], \rho^{-2n}\lambda] \\ & = & [\lambda, \rho^{-n}, \rho^{-2n}\lambda] \\ & \equiv & [\lambda, \rho^{-n}, \lambda][\lambda, \rho^{-n}, \rho^{-2n}] \ mod\, \gamma_4(\mathcal{H}(n)), \end{array}$$

and so,

$$\begin{array}{rcl} [\lambda,\rho^{-n},\rho^{-2n}] & \equiv & [\lambda,\rho^{-n},\lambda]^{-1}\, mod\, \gamma_4(\mathcal{H}(n)), \\ [\lambda,\rho,\rho]^{2n^2} & \equiv & [\lambda,\rho,\lambda]^{n}\, mod\, \gamma_4(\mathcal{H}(n)). \end{array}$$

Now, since $[c,\lambda]=e$, we have

$$[c,\rho]^{2n^2} \equiv e \;\; mod \;\; \gamma_4(\mathcal{H}(n)).$$

Since λ commutes with c, it follows that $\dfrac{\gamma_3(\mathcal{H}(n))}{\gamma_4(\mathcal{H}(n))}$ is cyclic of exponent divisor of $2n^2$.

□

Theorem 3. *The group $\mathcal{H}(n)$ is residually "torsion-free, abelian-by-finite solvable" and is nonsolvable.*

Proof. First of all, we will prove that for all $k \geq 0$, $\dfrac{\mathcal{H}(n)}{\times_{n^k}\gamma_2(\mathcal{H}(n))}$ is solvable, torsion-free and that

$$\bigcap\{\times_{n^k}\gamma_2(\mathcal{H}(n)), \mid k \geq 1\} = \{e\}.$$

(i) We assert that, the quotient group $\dfrac{\times_{n^k}\gamma_2(\mathcal{H}(n))}{\times_{n^{k+1}}\gamma_2(\mathcal{H}(n))}$ is torsion-free for all $k \geq 0$.

We will consider only the case $k=0$ as the general case can be treated similarly. Since

$$\gamma_2(\mathcal{H}(n)) = (\times_n\gamma_2(\mathcal{H}(n))) < c^{\rho^i} | 0 \leq i \leq n-2 >,$$

given $\omega \in \gamma_2(\mathcal{H}(n))$, there exist integers $r_0, r_1, \cdots, r_{n-2}$ and $c_0, c_1, \cdots, c_{n-1}$ elements of $\gamma_2(\mathcal{H}(n))$ such that

$$\omega = \left(c_0(\lambda^{-1}\rho^2)^{r_0-r_1}, \cdots, c_i(\lambda^{-1}\rho^2)^{r_i-r_{i+1}}, \cdots, c_{n-2}(\lambda^{-1}\rho^2)^{r_{n-2}}, c_{n-1}(\lambda^{-1}\rho^2)^{-r_0}\right).$$

If there exists $\omega \in \gamma_2(\mathcal{H}(n))$ such that $\omega \notin \times_n\gamma_2(\mathcal{H}(n))$ and $\omega^r \in \times_n\gamma_2(\mathcal{H}(n))$ for some $r \in \mathbb{Z}$, then, since $\dfrac{\mathcal{H}(n)}{\gamma_2(\mathcal{H}(n))}$ is torsion-free, it follows from the description of ω given above, that for $0 \leq i \leq n-3$,

$$\begin{array}{rcl} (\lambda^{-1}\rho^2)^{-r_0 r}, (\lambda^{-1}\rho^2)^{r_{n-2}r}, (\lambda^{-1}\rho^2)^{(r_i-r_{i+1})r} & \in & \gamma_2(\mathcal{H}(n)), \\ r_0 r & = & r_{n-2}r = (r_i - r_{i+1})r = 0. \end{array}$$

This implies that $r_i r = 0$, for $0 \leq i \leq n-2$.

If $r \neq 0$ then $\omega \in \times_n \gamma_2(\mathcal{H}(n))$, contradicting our hypothesis. Hence, $r = 0$ and $\dfrac{\gamma_2(\mathcal{H}(n))}{\times_n \gamma_2(\mathcal{H}(n))}$ is torsion-free.

(ii) We assert that the quotient group $\dfrac{\mathcal{H}(n)}{\times_{n^k} \gamma_2(\mathcal{H}(n))}$, $k \geq 0$ is torsion-free.

The case $k = 0$ follows from Proposition 2. Let us suppose that the statement is true for $k > 0$. Since $\dfrac{\mathcal{H}(n)}{\times_{n^{k+1}} \gamma_2(\mathcal{H}(n))}$ has a normal torsion-free subgroup, $\dfrac{\times_{n^k} \gamma_2(\mathcal{H}(n))}{\times_{n^{k+1}} \gamma_2(\mathcal{H}(n))}$, and, by induction, the quotient $\dfrac{\mathcal{H}(n)}{\times_{n^k} \gamma_2(\mathcal{H}(n))}$ is torsion-free. It follows that $\dfrac{\mathcal{H}(n)}{\times_{n^{k+1}} \gamma_2(\mathcal{H}(n))}$ is torsion-free.

(iii) We assert that the quotient group $\dfrac{\mathcal{H}(n)}{\times_{n^k} \gamma_2(\mathcal{H}(n))}$ is abelian-by-finite and solvable.

Since $\mathcal{H}(n) \leq \left(\times_{n^k} \mathcal{H}(n)\right)(\wr_k C_n)$, where $\wr_k C_n$ is the k-th iterated wreath product of the cyclic group C_n of order n, it follows that $\dfrac{\mathcal{H}(n)}{\times_{n^k} \gamma_2(\mathcal{H}(n))}$ is a homomorphic image of the abelian-by-finite and solvable group $\times_{n^k} \dfrac{\mathcal{H}(n)}{\gamma_2(\mathcal{H}(n))}(\wr_k C_n)$, and therefore the assertion holds .

(iv) We assert that $\bigcap\limits_{k=0}^{\infty} \left(\times_{n^k} \gamma_2(\mathcal{H}(n))\right)$ is trivial.

Since $\times_{n^k} \gamma_2(\mathcal{H}(n)) \leq \mathcal{H}_k(n)$, where $\mathcal{H}_k(n)$ it represents the subgroup of $\mathcal{H}(n)$ that stabilizes the k-th level of vertices of the tree $\mathcal{T}_n$, it follows that $\bigcap\limits_{k=0}^{\infty} \left(\times_{n^k} \gamma_2(\mathcal{H}(n))\right) \subseteq \bigcap\limits_{k=0}^{\infty} \mathcal{H}_k(n) = \{e\}$.

With this we have verified that $\mathcal{H}(n)$ is residually " torsion-free, abelian-by-finite and solvable".

(v) We assert that $\mathcal{H}(n)$ is nonsolvable.

Clearly, the second derived group $\mathcal{H}''(n)$ of $\mathcal{H}(n)$ is nontrivial and it contains the normal closure of $[c_2, c] = ((c^{-1}, e, c, e \cdots, e), e, \cdots, e)$ in $\mathcal{H}(n)$, which we denote by $\hat{D}(\mathcal{H}'(n))$. It is clear that for all $k \geq 1, \mathcal{H}^{(k+1)}(n) \geq (\hat{D}(\mathcal{H}'(n)))^{(k-1)}$, which is a subdirect product of $\underset{n}{\times}\mathcal{H}^{(k)}(n)$. By induction on k, we conclude that $\mathcal{H}(n)$ is nonsolvable.

□

Theorem 4. *The group $\mathcal{H}(n)$ is just-nonsolvable.*

Proof. It is sufficient show that for each $\omega \neq e$ in $\mathcal{H}(n)$, the normal closure of ω in $\mathcal{H}(n)$ contains some element of the derived series of $\mathcal{H}(n)$.

We consider two cases depending on whether or not $\omega \in \mathcal{H}_1(n)$. Let W be the normal closure of ω in $\mathcal{H}(n)$.

Case 1: ω active.

Let $\omega = (\omega_0, \omega_1, \cdots, \omega_{n-1})\sigma^i$, with $1 \leq i \leq n-1$. Then,

$$\begin{aligned}
[\omega, \lambda] &= (\omega_{n-i}^{-1}, \cdots, \underbrace{\omega_{n-1}^{-1}}_{i-1}, \omega_0^{-1}, \cdots, \omega_{n-(i+1)}^{-1})\sigma^{-i} \cdot (e, \cdots, e, \rho^{-1}\lambda\rho^{-1}) \cdot \\
&\quad (\omega_0, \omega_1, \cdots, \omega_{n-1})\sigma^i \cdot (e, \cdots, e, \rho\lambda^{-1}\rho) \\
&= (\omega_{n-i}^{-1}, \cdots, \underbrace{\omega_{n-1}^{-1}}_{i-1}, \omega_0^{-1}, \cdots, \omega_{n-(i+1)}^{-1}) \cdot (e, \cdots, e, \underbrace{\rho^{-1}\lambda\rho^{-1}}_{i-1}, e, \cdots, e) \cdot \\
&\quad (\omega_{n-i}, \omega_{n-(i-1)}, \cdots, \underbrace{\omega_{n-1}}_{i-1}, \omega_0, \cdots, \omega_{n-(i+1)}) \cdot (e, \cdots, e, \rho\lambda^{-1}\rho) \\
&= (e, \cdots, e, \underbrace{\omega_{n-1}^{-1}\rho^{-1}\lambda\rho^{-1}\omega_{n-1}}_{i-1}, e, \cdots, e, \rho\lambda^{-1}\rho).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
[\omega, \lambda, \lambda_n] &= [(e, \cdots, e, \underbrace{\omega_{n-1}^{-1}\rho^{-1}\lambda\rho^{-1}\omega_{n-1}}_{i-1}, e, \cdots, e, \rho\lambda^{-1}\rho), (e, \cdots, e, (\rho\lambda^{-1}\rho)^\rho)] \\
&= (e, \cdots, e, [\rho\lambda^{-1}\rho, \lambda^{-1}\rho^2]) \\
&= (e, \cdots, e, \rho^{-1}(\lambda\rho^{-3}\lambda\rho\lambda^{-1}\rho\lambda^{-1}\rho)^\rho) \\
&= \left(e, \cdots, e, (\lambda_0\lambda_3\lambda_2^{-1}\lambda_1^{-1})^\rho\right).
\end{aligned}$$

Let $\nu = \lambda_0\lambda_3\lambda_2^{-1}\lambda_1^{-1}$ and let N be the normal closure of ν in $\mathcal{H}(n)$. Then we have in Λ modulo N, $\lambda_3 \equiv \lambda_0^{-1}\lambda_1\lambda_2$ and, for $n \geq 3$, $\dfrac{\Lambda}{N}$ is an abelian group of rank at the most 3.

Thus $\dfrac{\mathcal{H}(n)}{N}$ is metabelian; that is, $\mathcal{H}''(n) \leq N$. Since the normal closure of $(e, \cdots, e, \nu)$ in $\mathcal{H}_1(n)$ is $\{e\} \times \cdots \times N$, it follows that $\underset{n}{\times} N$ is a subgroup of W, and therefore $\underset{n}{\times}\mathcal{H}''(n)$ is a subgroup of W.

Since $\dfrac{\mathcal{H}(n)}{\underset{n}{\times}\mathcal{H}''(n)}$ is a homomorphic image of the solvable group $\underset{n}{\times}\dfrac{\mathcal{H}(n)}{\mathcal{H}''(n)} < (\underset{n}{\times}\mathcal{H}''(n))\sigma >$ it follows that $\dfrac{\mathcal{H}(n)}{W}$ is solvable. Hence, there exists $k \geq 1$, such that $\mathcal{H}^{(k)}(n) \leq W$.

Case 2: ω inactive.

Let $\omega = (\omega_0, \omega_1, \cdots, \omega_{n-1})$. Without lost generality, suppose $\omega_{n-1} \neq 0$. Let W_{n-1} be the normal closure of ω_{n-1} in $\mathcal{H}(n)$. As the centralizer of $\mathcal{H}'(n)$ in $\mathcal{H}(n)$ is trivial, $[\omega_{n-1}, \mathcal{H}'(n)]$ is nontrivial.

Therefore,

$$W \geq [\omega, (e, \cdots, e, \mathcal{H}(n)')] = (e, \cdots, e, [\omega_{n-1}, \mathcal{H}'(n)]),$$

which implies

$$W \geq (e, \cdots, e, [W_{n-1}, \mathcal{H}'(n)]).$$

Since ω_{n-1} is nontrivial, there exists a minimum level where ω_{n-1} is active. At this level, the normal closure of ω_{n-1} in $\mathcal{H}(n)$ contains an element of the derived series of $\mathcal{H}(n)$, as in the first case. Therefore, W_{n-1} contains $\mathcal{H}^{(l)}(n)$ for some l and W contains $\underset{n}{\times}\mathcal{H}^{(l+1)}(n)$. As $\dfrac{\mathcal{H}(n)}{\underset{n}{\times}\mathcal{H}^{(l+1)}(n)}$ is solvable, the result is follows.

□

Let $O^{\{2n\}'}(\mathcal{H}(n))$ denote the residual subgroup

$$\bigcap \{K \trianglelefteq \mathcal{H}(n) \mid \mathcal{H}(n)/K \text{ has order coprime with } 2n\}.$$

Theorem 5. *The subgroup* $\gamma_3(\mathcal{H}(n)) = O^{\{2n\}'}(\mathcal{H}(n))$.

Proof. Let m be an integer relatively prime with $2n$ and let N be the normal closure of ρ^m in $\mathcal{H}(n)$. We will show that $\gamma_3(\mathcal{H}(n)) \leq N$.

In fact, in $\mathcal{H}(n)$ modulo N, $\lambda_{rm} = \lambda_0$ and $\Lambda_0 = \Lambda_{rm}$, for all $r \in \mathbb{Z}$. Since m and n are relatively prime, for each integer i, there exists an integer r_i such that $r_i m \equiv i \; mod(n)$. Therefore, $\Lambda_0 = \Lambda_i$ for all i. Furthermore, since Λ_i commutes with Λ_j, for all $i \not\equiv j \; mod(n)$, it follows that, modulo N, Λ is an abelian group generated by λ_0, λ_n and λ_{2n}. On viewing Λ as a $< \rho >$-module, we may re-write the equations $\lambda_m = \lambda_0$ and $\lambda_0^{-1}\lambda_n\lambda_{2n}\lambda_{3n}^{-1} = e$ in additive notation as

$$\lambda_0(\rho^m - 1) = 0, \lambda_0(-1 + \rho^n + \rho^{2n} - \rho^{3n}) = 0.$$

Thus ρ, in its action on Λ, satifies the equations

$$\begin{aligned} l(x) &= x^{3n} - x^{2n} - x^n + 1 = (x^n - 1)^2(x^n + 1) = 0, \\ r(x) &= x^m - 1 = (x - 1)r_1(x) = 0, \end{aligned}$$

where $r_1(x) = x^{m-1} + x^{m-2} + \cdots + x + 1$. Let GCD be the greatest common divisor operator applied to polynomials with integer coefficients. Then, since

$$\begin{aligned} GCD(x^n + 1, r_1(x)) &= 1, \; GCD(x^{n-1} + \cdots + x + 1, r_1(x)) = 1, \\ GCD(x - 1, r_1(x)) &= m, \end{aligned}$$

we conclude that

$$GCD(l(x), r(x)) = m(x - 1). \tag{1}$$

Let $a(x), b(x)$ be polynomials with integers coefficients such that

$$a(x)\left[(x^{n-1} + \cdots + x + 1)^2(x^n + 1)\right] + b(x)r_1(x) = 1. \tag{2}$$

Then, on multiplying the above equation by $(x - 1)^2$, we obtain

$$a(x)l(x) + b(x)(x - 1)r(x) = (x - 1)^2.$$

Since ρ is a root of $l(x)$ and of $r(x)$, using equations (1) and (2), the following equations hold:

$$\lambda_0 m(\rho - 1) = 0 \text{ and } \lambda_0(\rho - 1)^2 = 0,$$

in $\mathcal{H}(n)$ modulo N. Returning to multiplicative notation, we have

$$N\lambda_0^m = N\lambda_n^m = N\lambda_{2n}^m \;\text{ and }\; \lambda_2\lambda_1^{-2}\lambda_0 \in N.$$

Now, on developing $[c, \rho]$ in terms of ρ-conjugates of λ, we observe that $[c, \rho] = \lambda_2 \lambda_1^{-2} \lambda_0$. It follows then that $\gamma_3(\mathcal{H}(n)) \leq N$.

Hence,

$$\gamma_3(\mathcal{H}(n)) \leq O^{\{2n\}'}(\mathcal{H}(n)).$$

On the other hand, $\dfrac{\mathcal{H}^m(n)}{\gamma_3(\mathcal{H}(n))}$ is isomorphic to $< \rho^m, \lambda^m, c^m >$; therefore,

$$\bigcap_m \left\{ \frac{\mathcal{H}^m(n)}{\gamma_3(\mathcal{H}(n))} | \; gcd(m, 2n) = 1 \right\}$$

is isomorphic to

$$\bigcap_m < \rho^m, \lambda^m, c^m >= \{e\}.$$

We have reached $\bigcap_m \{\mathcal{H}(n)^m | \;\; \text{gcd } (m, 2n) = 1\} \leq \gamma_3(\mathcal{H}(n))$.

$\square$

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Said Sidki
Departamento de Matemática
Universidade de Brasília
70.910-900 - Brasília - DF
Brasil

Edméia F. da Silva
Universidade Federal de Goiás
Instituto de Matemática e Estatítica
Campus Samambaia - Caixa Postal 131
74.001-970 - Goiânia - GO - Brasil