ON THE SYMMETRIC AND REES ALGEBRAS OF (n,k)-CYCLIC IDEALS

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Abstract

Ordinary graph ideals were introduced and studied by Villarreal, Simis and Vasconcelos. Some higher order edge ideals have been introduced by Conca and De Negri [4], they have shown that if Γ is a tree then the ideal generated by all the square free monomial $X_{i_1}\dots X_{i_s}$ such that $\{x_{i_1},\dots,x_{i_s}\}$ is a path in Γ is a normal ideal of linear type. We prove that $\dim_{Krull} S(I) = \dim_{Krull} \mathcal{R}(I)$ when Γ is a cycle and deduce whether the ideal I is of linear type or not in many cases.

1 Introduction

Let $R = F[X_1,\dots,X_n]$ be a polynomial ring over a field F and I an ideal of R. We denote by $S(I)$ and by $\mathcal{R}(I)$ the symmetric algebra and the Rees algebra of I, respectively. Suppose that Γ is a graph with vertex set $\underline{X} = \{X_1,\dots,X_n\}$ and edges E. In [10], Villarreal defined the graph ideal $I(\Gamma)$ as the ideal generated by the monomials of the form X_iX_j where $(X_i,X_j) \in E$. Later, Conca and De Negri [2], generalized this notion considering the ideal $I_k(\Gamma)$ generated by all monomials $X_{i_1}\dots X_{i_k}$ such that $X_{i_1},\dots,X_{i_k}$ is a path in Γ. Further they showed that if Γ is a tree then $I_k(\Gamma)$ is of linear type and $\mathcal{R}(I_k(\Gamma))$ is normal and Cohen–Macaulay. For ordinary graph ideal of trees (e.g. when $t=2$) these results were already known in [8].

Let the graph Γ be a cycle of length n, and let $1 \le k \le n$. The main purpose of this article is to determine the lengths k of the the paths in Γ such that the ideal $I_k(\Gamma)$ is of linear type.

We answer the above question for the following cases:

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1) If $k = n-1$ then $I_{n-1}(\Gamma)$ is of linear type and $S(I_{n-1}(\Gamma))$ is Cohen-Macaulay (Theorem 3.2).

2) If n is odd and $k = n-2$ or $k = \frac{n-1}{2}$ then $I_k(\Gamma)$ is of linear type and $S(I_k(\Gamma))$ is Cohen-Macaulay (Theorem 3.4 and Corollary 3.6) .

3) If $gcd(n,k) = r > 1$ then $I_k(\Gamma)$ is not of linear type (Theorem 3.7).

4) If $gcd(n,k) = 1$ and $[\frac{n-1}{2}] < k \leq n-3$ then $I_k(\Gamma)$ is not of linear type (Theorem 3.8).

5) If $gcd(n,k) = 1$, $kl \equiv 1 \pmod{n}$ and $1 < k, l \leq [\frac{n-1}{2}]$ then $I_k(\Gamma)$ is not of linear type (Theorem 3.9a).

6) If $gcd(n,k) = 1$, $kl \equiv 1 \pmod{n}$, $1 < k \leq [\frac{n-1}{2}]$,$[\frac{n-1}{2}] < l < n$ and $n = t(n-l)+a$, $2 \leq a < n-l$ then $I_k(\Gamma)$ is not of linear type (Theorem 3.9b).

In Section 2, we define the (n,k)-cyclic ideal I_k, and we give a linear presentation of it when $k \geq \left[\frac{n}{2}\right]$. In the same section we use a convenient monomial order on $R[T_1, ..., T_n]$ in order to show that the Krull dimension of $S(I_k)$, $1 \leq k \leq n$, equals $n+1$ and so it coincides with the Krull dimension of $\mathcal{R}(I_k)$.

2 Notation and Preliminary results

Let us introduce some terminology and fix the notation that will be used throughout. $R = F[X_1, \ldots, X_n]$ will be a polynomial ring over a fixed field F. Let $I = (g_1, \ldots, g_p)$ be an ideal of R.

If I has a free presentation

$$R^m \xrightarrow{\gamma} R^p \xrightarrow{\beta} I \longrightarrow 0,$$

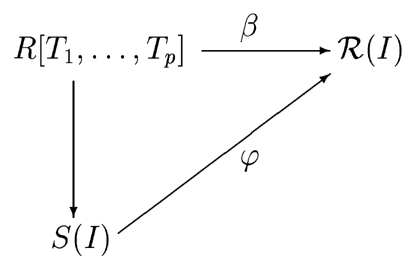
$\gamma = (a_{ij})$ then the symmetric algebra $S(I)$ of I is the quotient $R[T_1, \ldots, T_p]/q_\gamma(I)$. Here $R[T_1, \ldots, T_p]$ is a polynomial ring over R and $q_\gamma(I)$ is its ideal generated by the 1-forms

$$F_j = \sum a_{ij} T_i, \qquad 1 \leq j \leq m.$$

The Rees algebra $\mathcal{R}(I)$ of I is the subring $R[g_1T, \dots, g_pT]$ of $R[T]$ and hence we obtain an R-map

$$\beta\colon R[T_1, \dots, T_p] \to \mathcal{R}(I) \quad \text{defined by} \quad T_i \mapsto g_iT.$$

The kernel $q_\infty(I)$ of this map is generated by all forms $F(T_1, \dots, T_p)$ such that $F(g_1, \dots, g_p) = 0$. In particular $q_\gamma(I) \subseteq q_\infty(I)$ and there exist a canonical surjection $\varphi : S(I) \to \mathcal{R}(I)$ such that the following diagram is commutative:



The ideal I is of linear type if φ is an R-isomorphism of rings (ie, $q_\gamma(I) = q_\infty(I)$).

Definition 2.1 *The ideal $I_k \subset R$ is (n,k)-cyclic, $1 \le k \le n$, if I_k is generated by the set of square free monomials M_i, $i = 1, \dots, n$, $M_i = \prod_{v=1}^{k} X_{i+v}$. Here and in the sequel the indices are considered in $\mathbb{Z}_n$.*

Remark 2.2 *a) We could define I_k even for $k > n$. In this case $I_k = X.I_{n-k}$, where $X = X_1 \dots X_n$ according to the definition. This way, $S(I_k) \simeq S(I_{n-k})$ and $\mathcal{R}(I_k) \simeq \mathcal{R}(I_{n-k})$. As we can see, $k \le n$ i s not a real restriction.*

b) Notice that if $k < n$ then the minimal number of generators of I is n.

Assume that $1 \le k \le n-1$ and let $I_k = (M_1, \dots, M_n)$ be the (n,k)-cyclic ideal. Consider $\beta\colon R^n \to I_k$ defined by $\beta(e_i) = M_i$ where $\{e_1, \dots, e_n\}$ is the canonical basis of R^n. It is known that $\ker\beta$ is generated by

$$\sigma_{ij} = \frac{M_i}{d_{ij}} \cdot e_j - \frac{M_j}{d_{ij}} \cdot e_i, \qquad d_{ij} = \gcd(M_i, M_j).$$

Then we have the following proposition:

Proposition 2.3 *The ideal I_k admits the free presentation*

$$R^m \xrightarrow{\overline{\alpha}} R^n \xrightarrow{\beta} I_k \longrightarrow 0.$$

Here $m = \binom{n}{2}$, $\overline{\alpha}$ is an $n \times m$ matrix given by the relations σ_{ij}, and $\alpha_{n\times n}$ is the submatrix of $\overline{\alpha}$,

$$\alpha = \begin{pmatrix} -X_{k+2} & 0 & 0 & \dots & 0 & X_1 \\ X_2 & -X_{k+3} & 0 & \dots & 0 & 0 \\ 0 & X_3 & -X_{k+4} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -X_{k+n} & 0 \\ 0 & 0 & 0 & \dots & X_n & -X_{k+1} \end{pmatrix}.$$

Proof. The existence of this free presentation was justified above. Since

$$\sigma_{(j-1)j} = X_j \cdot e_j - X_{j+k} \cdot e_{j-1}$$

for every j, one can rearrange the indices in an appropriate manner in order to obtain α as the submatrix of $\overline{\alpha}$ formed by its first n columns.

□

If $M_i = \prod_{v=1}^{k} X_{i+v}$, we denote $supp(M_i) = \{X_{i+1}, \cdots, X_{i+k}\}$ and let N be the submodule of R^n generated by the set $\{\sigma_{(j-1)j} \mid j \in \mathbb{Z}\}$. Recall that the indices are considered modulo n.

Lemma 2.4 *Assume that i and j, $1 \leq i < j \leq n$, satisfy one of the conditions:*

(i) $supp(M_i) \cap supp(M_j) \neq \phi$,

(ii) $n = 2k$ and $supp(M_i) \cap supp(M_j) = \phi$, or

(iii) $n = 2k+1$ and $supp(M_i) \cap supp(M_j) = \phi$.

Then $\sigma_{ij} \in N$.

Proof. If $a \in \mathbb{Z}$ we define the F-automorphism (of rings) $\varphi_a : R \to R$ by $\varphi_a(X_i) = X_{i+a}$. We will also denote by φ_a the F-isomorphism of F-vector spaces $\varphi_a : R^n \to R^n$ given by

$$\varphi_a(\sum_{i=1}^{n} f_i e_i) = \sum_{i=1}^{n} \varphi_a(f_i).e_{i+a}.$$

Notice that $\varphi_a(\sigma_{ij}) = \sigma_{(i+a)(j+a)}$, therefore $\varphi_a(N) = N$ and thus we can suppose $i = 1$.

Since $X_j e_j \equiv X_{j+k} e_{j-1} \pmod N$ it is easy to see by induction over $j - r$, $1 \leq r \leq j$, in the first case and by induction over r in the second, that

$$X_r \ldots X_j.e_j \equiv X_{r+k} \ldots X_{j+k}.e_{r-1} \pmod N \qquad (*)$$

and

$$X_{r+1} \ldots X_{r+n-k}.e_{r+n-k} \equiv X_1 \ldots X_{n-k}.e_{n-k} \pmod N \qquad (**)$$

In particular, when $r = k + 1$ we have:

$$X_{k+2} \ldots X_n \cdot X_1.e_{n+1} \equiv X_1 \ldots X_{n-k}.e_{n-k} \pmod N \qquad (**)$$

In order to prove i), one observes that if $supp(M_1) \cap supp(M_j) \neq \phi$ then:

a) $\sigma_{1j} = X_2 \cdots X_j \cdot e_j - X_{k+2} \cdots X_{j+k} \cdot e_1$, if $j \leq k$ and $j + k \leq n + 1$.

b) $\sigma_{1j} = X_{u+1} \cdots X_j \cdot e_j - X_{k+2} \cdots X_n \cdot X_1 \cdot e_1$, if $j \leq k$, $n + 2 \leq j + k$ and $u = j + k - n$.

c) $\sigma_{1j} = X_{u+1} \cdots X_j \cdot e_j - X_{j+1} \cdots X_n \cdot X_{n+1} \cdot e_1$, if $k < j$, $n + 2 \leq j + k$ and $u = j + k - n$.

In case a), one sets $r = 2$ in (*) and obtains that $\sigma_{1j} \in N$. In c) one sets $r = j + 1$ in (*), hence $\sigma_{1j} \in N$. In case b) use that $j = u + n - k$ and it follows from (**) that $\sigma_{1j} \in N$.

The assertions ii) and iii) are dealt with in analogous way.

□

Proposition 2.5 *Let $I_k \subset R$ be (n,k)-cyclic ideal and $k \geq \left[\frac{n}{2}\right]$. Then I_k has the following free presentation*

$$R^n \xrightarrow{\alpha} R^n \xrightarrow{\beta} I \longrightarrow 0,$$

where $\beta(e_i) = M_i$ and

$$\alpha = \begin{pmatrix} -X_{k+2} & 0 & 0 & \dots & 0 & X_1 \\ X_2 & -X_{k+3} & 0 & \dots & 0 & 0 \\ 0 & X_3 & -X_{k+4} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -X_{k+n} & 0 \\ 0 & 0 & 0 & \dots & X_n & -X_{k+1} \end{pmatrix}.$$

Proof. According to the assumption $k \geq \left[\frac{n}{2}\right]$ we have that some of the conditions of Lemma 2.4 is satisfied. Then Lemma 2.4 implies that Kerβ=N where N is generated by $\{\sigma_{(j-1)j} \colon j \in \mathbb{Z}\}$.

□

We assume from now on that $\overline{R} = R[T_1, \dots, T_n]$, and if $1 \leq k < n$, $\overline{\alpha}$ is the matrix defined in Proposition 2.3. We denote $q_k := q_{\overline{\alpha}}(I_k) \subseteq \overline{R}$, $q_\infty(k) = q_\infty(I_k) \subseteq \overline{R}$ and $\overline{q}_k = (g_1, ..., g_n)$, where $g_j = X_{j+1}T_{j+1} - X_{j+1+k}T_j$ for all j. Notice that $\overline{q}_k \subseteq q_k \subseteq q_\infty(k)$ and by the last Proposition $\overline{q}_k = q_k$ if $n > k \geq \left[\frac{n}{2}\right]$.

In what follows we use the reverse lexicographic order on $\overline{R}$ induced by the following order on the variables

$$X_n > \ldots > X_1 > T_n > \ldots > T_1.$$

If $f \in \overline{R}$ we denote by $in(f)$ the initial term of f with respect to this order. If f, $g \in \overline{R}$ we denote by $S(f,g)$ the S-polynomial of f and g.

Proposition 2.6 *Let I_k be the (n,k)-cyclic ideal. Then $dim S(I_k) = n+1 = dim \mathcal{R}(I_k)$.*

Proof. Since R is a domain of dimension n, $\dim\mathcal{R}(I_k) = n+1$, so $\dim S(I_k) \geq n+1$.

If $k = n$ we have that $I_n = (X_1 \cdots X_n)$ i.e., I_n is a free module of rank 1 and then $S(I_n) \simeq \mathcal{R}(I_n) \simeq R[T]$.

On the other hand, if $1 \leq k < n$ we know that $g = (g_1, \ldots, g_{n-1}) \subset \overline{q}_k \subseteq q_k$ and $\{g_1, \ldots, g_{n-1}\}$ is a regular sequence on $\overline{R}$, because $\{X_2T_2, \ldots, X_nT_n\}$ is a regular sequence and $X_{j+1}T_{j+1} = in(g_j), j = 1, ..., n-1$ [3,Proposition 15.15]. This way we have height $(q_k) \geq n-1$ and $\dim S(I_k) \leq n+1$.

□

3 Are $(n,k)-$cyclic ideals of linear type?

The main purpose of this section is to determine pairs $(n,k), 1 \leq k \leq n$ that we can decide if I_k is of linear type or not.

If $k = n$ then I_n is a free R-module of rank 1 and $S(I_n) \simeq \mathcal{R}(I_k) \simeq R[T]$. Hence I_k is the linear type and $S(I_k)$ is a regular ring.

When $k = 1$ we have that $I_1 = (X_1, \ldots, X_n)$ is generated by a regular sequence. Thus is follows from [3] that it is of linear type.

In the case n is odd and $k = 2$, it was proved by Villarreal in [10] that I_2 is of linear type.

Hence, till the end of this section, we will suppose that $n \geq 5$ and $3 \leq k \leq n-1$. For readers' convenience we summarize the contents of the section.

1) If $k = n-1$ then I_{n-1} is of linear type and $S(I_{n-1})$ is Cohen-Macaulay.

2) If n is odd and $k = n-2$ or $k = \frac{n-1}{2}$ then I_k is of linear type and $S(I_k)$ is Cohen-Macaulay.

3) If $gcd(n,k) = r > 1$ then I_k is not of linear type.

4) If $gcd(n,k) = 1$ and $[\frac{n-1}{2}] < k \leq n-3$ then I_k is not of linear type.

5) If $gcd(n,k) = 1$, $kl \equiv 1 \pmod{n}$, and $1 < k, l \leq \left[\frac{n-1}{2}\right]$ then I_k is not of linear type.

6) If $gcd(n,k) = 1$, $kl \equiv 1 \pmod n$, $1 < k \leq [\frac{n-1}{2}], [\frac{n-1}{2}] < l < n$ and $n = t(n-l) + a$, $2 \leq a < n-l$ then $I_k(\Gamma)$ is not of linear type.

Using the notation adopted we have $S(I_k) \simeq \overline{R}/q_k$. From Proposition 2.3 $\overline{q}_k \subseteq q_k$ and further if $k \geq \left[\frac{n}{2}\right]$, according to Proposition 2.5, we have that $\overline{q}_k = q_k$. It is well-known that I_k is of linear type if and only if q_k is a prime ideal. Therefore we have to verify whether q_k is prime or not.

Let $\overline{\alpha}$ be the matrix given by the Proposition 2.3 and $1 \leq t \leq n$, the ideal generated by the $t \times t$ minors of $\overline{\alpha}$ will be denoted by $I_t(\overline{\alpha})$. With this notation we have:

Lemma 3.1 *Let $I_k \subset R$ be (n,k)-cyclic ideal, $1 \leq k \leq n-1$. Then $n-1 = rank(\overline{\alpha})$ and $height(I_t(\overline{\alpha})) \geq n-t+1$ for all t, $1 \leq t \leq n-1$, i.e., I_k satisfies $\mathfrak{F}_1$.*

Proof. Since $rank(I_k) = 1$ and $R^m \xrightarrow{\overline{\alpha}} R^n \xrightarrow{\beta} I_k \longrightarrow 0$, we have that $n-1 = rank(\overline{\alpha})$.

Now, by Proposition 2.3 we know that α, given by Proposition 2.5, is a submatrix of $\overline{\alpha}$, hence $I_t(\alpha) \subseteq I_t(\overline{\alpha})$. It is easy to see that for every t, $1 \leq t \leq n-1$, $I_t(\alpha)$ contains the ideal J_t where J_t is the ideal generated by all square free monomials of degree t. Further it is well know by the theory of Stanley-Reisner rings that $height(J_t) = n-t+1$. Thus the lemma is proved.

□

Theorem 3.2 *If $k = n-1$ then I_k is of linear type and $S(I_k)$ is Cohen-Macaulay.*

Proof. In this case

$$\overline{q}_{n-1} = q_{n-1} = (X_2T_2 - X_1T_1, \ldots, X_nT_n - X_{n-1}T_{n-1}, X_1T_1 - X_nT_n)$$

and it is easy to see that $q_{n-1} = (f_1, \ldots, f_{n-1})$ where $f_j = X_jT_j - X_1T_1$. In other words we have the following presentation for I_{n-1}:

$$R^{n-1} \xrightarrow{\alpha'} R^n \longrightarrow I_{n-1} \longrightarrow 0,$$

where α' is the $n \times (n-1)$ matrix

$$\alpha' = \begin{pmatrix} -X_1 & -X_1 & \dots & -X_1 \\ X_2 & 0 & \dots & 0 \\ 0 & X_3 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & X_n \end{pmatrix}.$$

Since $rank(\alpha') = n-1$ then I_{n-1} is an ideal of projective dimension 1. But we know that if $height(I_t(\alpha')) \geq n-t+1$, $1 \leq t \leq n-1$, thus by [4, Theorem 1.1] we obtain that I_{n-1} is of linear type and $S(I_{n-1})$ is Cohen-Macaulay.

□

Now assume that $\left[\frac{n-1}{2}\right] \leq k \leq n-2$ and denote $r = n-k-1$. Consider $I_r = (\tilde{M}_1, \dots, \tilde{M}_n)$ where $\tilde{M}_j = X_{j+1} \dots X_{j+r}$. Let ψ_r be the column matrix given by

$$\psi_r = \begin{pmatrix} \tilde{M}_{k+2} \\ \tilde{M}_{k+3} \\ \vdots \\ \tilde{M}_{k+n} \\ \tilde{M}_{k+n+1} \end{pmatrix} = \begin{pmatrix} \tilde{M}_{k+2} \\ \tilde{M}_{k+3} \\ \vdots \\ \tilde{M}_{k} \\ \tilde{M}_{k+1} \end{pmatrix}.$$

Proposition 3.3 *If* $\left[\frac{n-1}{2}\right] \leq k \leq n-2$ *and* $r = n-k-1$ *then*

$$0 \longrightarrow R \xrightarrow{\psi_r} R^n \xrightarrow{\alpha} R^n \xrightarrow{\beta} I_k \longrightarrow 0$$

is a minimal free resolution of I_k. *Here* α *is the matrix of the presentation given by Proposition 2.5.*

Proof. A direct verification shows that $\alpha\psi_r = 0$. Notice that $rank(\alpha) = n-1$ and $grade(I_1(\psi_r)) \geq 2$ since $r \leq \frac{n}{2}$. Hence by the acyclicity criterion of Buchsbaum and Eisenbud we have that the complex

$$0 \longrightarrow R \xrightarrow{\psi_r} R^n \xrightarrow{\alpha} R^n \xrightarrow{\beta} I_k \longrightarrow 0$$

is a free resolution of I_k.

This resolution is minimal because $\psi_r(R) \subseteq \mathbf{m}R^n$ and $\alpha(R^n) \subseteq \mathbf{m}R^n$ where $\mathbf{m} = (X_1, \dots, X_n)$.

□

Theorem 3.4 *If n is odd and $k = n - 2$ then I_k is of linear type and $S(I_k)$ is Cohen–Macaulay.*

Proof. Observe that if $k = n - 2$ then $r = n - k - 1 = 1$ and the entries of the column ψ_1 form a regular sequence. Since I_{n-2} satisfies $\mathcal{F}_1$, by Theorem 7.5.1 of [9] and by Proposition 3.3 we have that $(I_{n-2})_{\mathbf{m}}$, where $\mathbf{m} = (X_1, \ldots, X_n)$, is of linear type and $S((I_{n-2})_{\mathbf{m}}) = \frac{R_{\mathbf{m}}[T_1,\ldots,T_n]}{(\overline{q}_k)_{\mathbf{m}}}$ is Cohen-Macaulay. Then $(S(I_{n-2}))_{\overline{\mathbf{m}}}$ with $\overline{\mathbf{m}} = (X_1, \ldots, X_n, T_1, \ldots, T_n)$ is a Cohen-Ma caulay domain, and by [1,Corollary 2.2.15], it follows that $S(I_{n-2})$ is Cohen-Macaulay domain. □

Now consider the case $3 \leq k \leq n - 2$ with $\gcd(k, n) = 1$. Let $2 \leq l \leq n - 2$ be such that $lk \equiv 1 \pmod{n}$ and set $l' = n - l$. Define the F-automorphism δ_k of the ring $\overline{R} = R[T_1, \ldots, T_n]$ by $\delta_k(X_i) = T_{il'}$ and $\delta_k(T_i) = X_{il'}$. Let $k' = n - k$ and observe that $\delta_k^{-1} = \delta_l$, where $\delta_l(X_i) = T_{ik'}$ and $\delta_l(T_i) = X_{ik'}$.

Lemma 3.5 *If $3 \leq k \leq n - 2$ and $\gcd(n, k) = 1$ then:*

a) $\delta_k(\overline{q}_k) = \overline{q}_l$

b) $\delta_k(q_\infty(k)) = q_\infty(l)$ *and* $\mathcal{R}(I_k) \simeq \mathcal{R}(I_l)$

Proof. a) Since $\delta_k^{-1} = \delta_l$ it is sufficient to observe that

$$\delta_k(X_iT_i - X_{i+k}T_{i-1}) = X_{n-il}T_{n-il} - X_{(n-il)+l}T_{(n-il)-1}.$$

b) Let $B = \overline{R}[t]$ be a polinomial ring over $\overline{R}$, it is known that $q_\infty(k) = J \cap \overline{R}$, where $J = (T_1 - tM_1, \ldots, T_n - tM_n) \subset B$. J is a binomial ideal and by [3,corollary 3.1], $q_\infty(k)$ is a binomial ideal, ie, $q_\infty(k)$ is generated by polynomials of type $F = NT_{i_1}T_{i_2}\cdots T_{i_r} - LT_{j_1}\cdots T_{j_r}$, where N, L are monomials in R, $i_1 \leq i_2 \cdots \leq i_r$, $j_i \leq \cdots \leq j_r$ and $j_r \leq i_r$. For all i, $X_iT_i \equiv_{\overline{q}_k} X_{i+k}T_{i-1}$ and $q_\infty(k)$ is a prime ideal it is easy to show using induction over r and $i_r - j_r$ that $X^mF \in \overline{q}_k$ for some $m \in \mathbb{N}$, where $X = X_1 \ldots X_n$. So, if m is sufficiently great $X^m q_\infty(k) \subseteq \overline{q}_k$, but $\delta_k(X) = T = T_1 \cdots T_n$ and $\delta_k(\overline{q}_k) = \overline{q}_l$ then $T^m\delta_k(q_\infty(k)) \subseteq \overline{q}_l \subseteq q_\infty(l)$. Observe that $\delta_k(q_\infty(k))$ and $q_\infty(l)$ are prime ideals and they have the same height, so $\delta_k(q_\infty(k)) = q_\infty(l)$ and $\mathcal{R}(I_k) \simeq \mathcal{R}(I_l)$.

□

Corollary 3.6 *If n is odd and $l = \frac{n-1}{2}$ then I_l is of linear type and $S(I_l)$ is Cohen–Macaulay.*

Proof. Obviously $\frac{n-1}{2}(n-2) \equiv 1 \pmod{n}$, and by Lemma 3.2 we have that $\delta_k(q_k) = q_l$ where $k = n-2$. But $S(I_k) = \overline{R}/q_k$ and $S(I_l) = \overline{R}/q_l$ because $l = \frac{n-1}{2}$. Therefore $S(I_k) \simeq S(I_l)$ and by Theorem 4.2 we are done.

□

Theorem 3.7 *If $3 \leq k \leq n-2$ and $\gcd(k,n) = r > 1$ then I_k is not of linear type.*

Proof. If $X = X_1 \dots X_n$, $t = n/r$ and $s = k/r$ then

$$X^s = X_1^s \dots X_n^s = (X_1 \dots X_k)(X_{k+1} \dots X_{2k}) \dots (X_{(t-1)k+1} \dots X_{tk})$$

and we rewrite it in the following form

$$X^s = (X_2 \dots X_{k+1})(X_{k+2} \dots X_{2k+1}) \dots (X_{(t-1)k+2} \dots X_{tk} X_1).$$

Then $g = T_n T_k \dots T_{(t-1)k} - T_1 T_{k+1} \dots T_{(t-1)k+1}$ is some relation of the Rees algebra of I_k but it is not in the defining ideal of $S(I_k)$.

□

Our next goal is to deal with the case when $\gcd(n,k) = 1$ and $\left[\frac{n}{2}\right] \leq k \leq n-3$.

In what follows, we use the reverse lexicographic order on $\overline{R} = R[T_1, \dots, T_n]$ as defined in the second paragraph.

In [4, Proposition 1.10] Eisenbud and Sturmfels proved the following proposition.

Proposition [E–S] *Let B be a binomial ideal and let M be a monomial ideal in $\overline{R} = F[X_1, \dots, X_n, T_1, \dots, T_n]$. If $f \in B + M$ and f' is the sum of those terms of f that are not individually contained in $B + M$ then $f' \in B$.*

Theorem 3.8 *If* $\left[\frac{n}{2}\right] < k \leq n-3$ *with* $\gcd(n,k)=1$ *then* I_k *is not of linear type.*

Proof. We have that $S(I_k) \simeq \overline{R}/q_k$ where $q_k = (g_1, \dots, g_n)$,

$$g_i = X_{i+1}T_{i+1} - X_{i+1+k}T_i, \qquad 1 \leq i \leq n-2,$$

$g_n = X_{k+1}T_n - X_1T_1$ and $g_{n-1} = X_nT_n - X_kT_{n-1}$.

Set $h_1 = S(g_n, g_{n-1})$ and if $2 \leq i \leq k+1$, define by induction $h_i = S(h_{i-1}, g_{n-i})$. An easy induction argument shows that

$$h_i = (\prod_{w=w_0}^{k+1} X_w)T_{n-i} - X_1(\prod_{v=v_0}^{n} X_v)T_1$$

where $w_0 = k-i+1$ and $v_0 = n-i+1$.

Let t, $a \in \mathbb{N}$ be such that $n = t(n-k)+a$, $1 \leq a < n-k$. Since $\gcd(k,n)=1$ and $\left[\frac{n}{2}\right] < k \leq n-3$ we have that $t \geq 2$, $k-a = (t-1)(n-k)$ and $t(n-k) \geq k+1$.

Now we consider two cases.

First case. $t(n-k) \geq k+2$.

According to the expression for h_i we have that

$$h_a = (\prod_{w=w_1}^{k+1} X_w)T_{t(n-k)} - X_1(\prod_{v=1}^{a} X_{t(n-k)+v})T_1$$

where $w_1 = k-a+1 = (t-1)(n-k)+1$. Define $m_1 = S(h_a, g_{(t-1)(n-k)})$ and $m_s = S(m_{s-1}, g_{(t-s)(n-k)})$ for $2 \leq s \leq t-1$. Then by induction one obtains $m_{t-1} = X_1(U_1 - U_2) \in q_k$, where

$$U_1 = (\prod_{w=k-a+2}^{k+1} X_w)(\prod_{q=1}^{t} T_{q(n-k)}), \quad U_2 = (\prod_{v=1}^{a} X_{t(n-k)+v})(\prod_{q=1}^{t-1} T_{q(n-k)+1}).T_1.$$

If we suppose that q_k is a prime ideal then $U = U_1 - U_2 \in q_k$. But T_n and X_1 do not divide any term of U, then $U \in B + M$ where $B = (g_1, \dots, g_{n-2})$ and $M = X_kT_{n-1}.\overline{R}$.

Since $in(g_i) = X_iT_i$ for all i, $1 \leq i \leq n-2$ we have that $\{g_1, \dots, g_{n-2}\}$ is a Gröbner basis for B, because $\gcd(in(g_i), in(g_j)) = 1$, for $1 \leq i < j \leq n-2$.

Observe that T_{n-1} does not divide U_2, thus if $U_2 \in B+M$ we have that $U_2 \in B$. But this is impossible because X_iT_i does not divide U_2 for any $i = 1, 2, \ldots, n-2$.

Since $U = U_1 - U_2 \in B + M$ and $U_2 \notin B + M$ we have that $U_1 \notin B + M$. Thus, by Proposition E-S, $U \in B$. But $U_1 = in(U)$, therefore X_iT_i divides U_1 for some i, $1 \leq i \leq n-2$. Then there exists w, $(t-1)(n-k)+2 \leq w \leq k+1$, and s, $2\ les \leq t$ such that $w = s(n-k)$. Hence $s = t$ and $t(n-k) \leq k+1$ which is a contradiction with $t(n-k) \geq k+2$. Therefore $U \notin q_k$ and q_k is not a prime ideal.

Second case. $t(n-k) = k+1$.
Let $b = n-k-2$, then we have

$$h_b = (\prod_{w=(t-1)(n-k)+2}^{k+1} X_w).T_{k+2} - X_1(\prod_{v=k+3}^{n} X_v).T_1.$$

Set $H_1 = S(h_b, g_{w_1})$ where $w_1 = (t-1)(n-k)+1$, and define inductively $H_j = S(H_{j-1}, g_{(t-j)(n-k)+1})$ for $2 \leq j \leq t-1$. It is easy to see that

$$H_{t-1} = X_2(\prod_{w=w_1+2}^{k+1} X_w)(\prod_{r=1}^{t} T_{r(n-k)+1}) - X_1(\prod_{v=k+3}^{n} X_v).T_1(\prod_{r=1}^{t-1} T_{r(n-k)+2}).$$

Define $H = S(H_{t-1}, g_{n-k})$, then $H = X_1(W_1 - W_2) = X_1 W$ where

$$W_1 = X_2(\prod_{w=w_1+2}^{k+1} X_w).T_{n-k}.(\prod_{r=2}^{t} T_{r(n-k)+1}),$$

$$W_2 = X_{n-k+1}(\prod_{v=k+3}^{n} X_v).T_1.(\prod_{v=1}^{t-1} T_{v(n-k)+2}).$$

Now, arguments similar to those of the previous case show that $W \notin q_k$. Therefore q_k is not prime.

□

When $k \leq \left[\frac{n-1}{2}\right]$, by Proposition 2.3,we have that $q_k = \overline{q}_k + q'_k$, where q'_k is generated by binomials of degree $\geq k$ in the variables $X_1, \ldots, X_n$.

Theorem 3.9 *Let $k \geq 3$, $\gcd(n,k) = 1$ and $lk \equiv 1 \pmod{n}$, $1 \leq l < n$. Then I_k is not of linear type in the following cases:*

a) $1 < l, k \leq \left[\frac{n-1}{2}\right]$

b) $1 < k \leq \left[\frac{n-1}{2}\right]$, $\left[\frac{n-1}{2}\right] < l$ *and* $n = t(n-l) + a$, *with* $2 \leq a < n - l$

Proof. a) Here, we define $h_1 = S(g_n, g_k)$ and $h_i = S(h_{i-1}, g_{ik})$. It is not difficult to see that

$$h_i = X_{(i+1)k+1}(\prod_{v=0}^{i} T_{vk}) - X_1(\prod_{t=0}^{i} T_{tk+1}).$$

Thus, we have that for $i = l - 1$,

$$h_{l-1} = X_2(\prod_{v=0}^{l-1} T_{vk}) - X_1(\prod_{t=0}^{l-1} T_{tk+1}).$$

Set $H = S(h_{l-1}, g_1)$, hence $H = T_1 G$ where

$$G = X_{2+k} T_n T_k \dots T_{(l-1)k} - X_1 T_2 T_{k+1} \dots T_{(l-1)k+1}.$$

Now, if q_k is a prime ideal then $G \in q'_k = \overline{q}_k + q'_k$. But q'_k is generated by binomials of degree $\geq k$ in the variables $X_1, \dots, X_n$, and thus $G \in q_k$. Observe that T_1 and X_2 do not divide any term of G, therefore $G \in B' + M'$ where $B' = (g_2, \dots, g_{n-1})$, $M' = (X_{k+1} T_n)\overline{R}$ and $\{g_2, \dots, g_{n-1}\}$ is a Gröbner basis for B'.

On the other hand $2 \not\equiv bk \pmod{n}$ for every $0 \leq b \leq l - 1$, because $l \leq \left[\frac{n-1}{2}\right]$. Therefore $in(G) = G_1 = X_{2+k} T_n T_k \dots T_{(l-1)k}$.

Continuing as in the proof of the last theorem, we obtain that $G \in B'$. Hence there exists b, $1 \leq b \leq l - 1$ such that, $k + 2 \equiv bk \pmod{n}$. Then $1 + 2l \equiv b \pmod{n}$, but this is impossible since $2l < n$.Therefore q_k is not a prime ideal.

b) Since $k(n-l) \equiv -1 \pmod{n}$ and $n = t(n-l) + a$, with $2 \leq a < n-l$, we have $sn = k(n-l) + 1$ with $s > 1$. Then $(s-1)n + a - 1 = (k-t)(n-l)$ and so

$k > t$. Now, substituting k by l in Proposition 3.8 follows that the polynomial U (in the First case) o r the polynomial W (in the Second case) given in the proof of the propositon are not in $q_l = \overline{q}_l$. But U (or W) is a binomial of degree t in the variables $T_1, \ldots, T_n$ and so $\delta_l(U)$ (or $\delta_l(W)$) has degree t in the variables $X_1, \ldots, X_n$, where δ_l is the automorphism defined in Lemma 3.5. As $t < k$, $\delta_l(U) \in q_k$ if, and only if, $\delta_l(U) \in \overline{q}_k$, but $\delta_l(\overline{q_l}) = \overline{q}_k$ though $\delta_l(U)$ (or $\delta_l(W)$) is not in q_k, anyway, by Lemma 3.5 $\delta_l(U) \in q_\infty(k)$ (or $\delta_l(W) \in q_\infty(k)$).

□

Remark 3.10 *The only exception in our description is the case $n = k(n-l)+1$. We performed some tests using Macaulay software and obtained pairs (n,k) like $n = 16 = 3.5+1$ and $k = 2$ or $k = 5$ where I_k is not of linear type and, if $n = 10 = 3.3+1$ and $k = 3$, and in this case I_k is of linear type. From these results we can see that ideals (n,k) cyclic with $n = k(n-l)+1$ could be of linear type or not.*

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