INCOMPRESSIBLE FLOW IN ISOTROPIC GRANULAR POROUS MEDIA IN A TIME DEPENDENT DOMAIN

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Abstract

In this work we study the existence of weak solutions to Navier-Stokes type equations defined in a noncylindrical domain $\hat{\mathrm{Q}}$, where $\hat{\mathrm{Q}}$ is the image of a cylinder Q of $I\!R^{n+1}$ and $\hat{\mathrm{Q}}$ is not necessarily increasing or decreasing in time.

1 Introduction

In this work we will prove results concerning the existence of weak solutions of a system of partial differential equations corresponding to a generalization of the classical Navier-Stokes equations on a noncylindrical domain. The equations are the following:

$$\begin{cases} \rho u_t + \rho u \cdot \nabla \left(\dfrac{u}{\eta}\right) - \mu \Delta u + \eta \nabla p + \mu F(\eta)\, u = \rho \eta f \quad \text{in } \hat{Q}, \\ div\ u = 0 \quad \text{in } \hat{Q}, \\ u(x,0) = u_o(x) \ , \ \forall x \in \Omega_0, \\ u(x,t) = 0 \ , \ \forall t \in (0,T) \ , \ \forall x \in \hat{\Sigma}. \end{cases} \tag{PNC}$$

Let $T > 0$ be a real number and $\{\Omega_t\}, 0 \leq t \leq T$ a family of bounded open sets of $I\!R^n$ with boundary $\partial\Omega_t$. Let us consider the noncylindrical domain of $I\!R^{n+1}$

$$\hat{Q} = \bigcup_{0<t<T} \Omega_t \times \{t\} \quad \text{with lateral boundary} \quad \hat{\Sigma} = \bigcup_{0<t<T} \partial\Omega_t \times \{t\}.$$

The unknowns in the problem are $u(x,t) \in I\!R^n$ and $p(x,t) \in I\!R$, which denote, respectively, the fluid velocity and the hydrostatic pressure at a point

$x \in \Omega$, at time $t \in [0,T]$. We assume that the fluid viscosity, μ, is a positive constant. The density, ρ, without loss of generality, will be assumed to be normalized to be one. The porosity $\eta(x,t)$ at a point $x \in \Omega$, at a time $t \in [0,T]$, is defined in rough terms as the void volume divided by the total volume of small regions in the neighborhood of x at a time t. Thus, $\eta(x,t)$ assumes real values between zero and one. We observe that the porosity is one in cavities, where, therefore, the flow is free. At points (x,t) such that the porosity is zero, the material medium is purely solid and can be excluded from the flow region. Throughout this work, we will assume that the porosity satisfies $0 < \eta(x,t) \leq 1$. F is a force term due to the friction between the granular porous medium and the fluid. On physical grounds, F is a continuous function satisfying $\lim_{z\to 1} F(z) = 0$ and $\lim_{z\to 0} F(z) = \infty$ (see Prieur du Plessis and Masliyah in [8] for an expression for F.) We remark that our results will not depend on that particular expression for F. A known external force field, such as gravity, is denoted $g(x,t)$ and may be acting on the flow. In cartesian coordinates, we have

$$\Delta u = (\Delta u_1, \ldots, \Delta u_n) \quad \text{and} \quad (u \cdot \nabla u)_i = \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j}$$

Observe that the classical Navier-Stokes equations are a particular case of these equations when $\eta \equiv 1$. There exist several works for Navier-Stokes equations in noncylindrical domains, among them the works of J.L. Lions [6], R. Salvi [9] and, recently, M. Milla Miranda and J. Límaco Ferrel [7].

This work is organized as follows: for the next section (Preliminaries) we will present the notation, introduce the many functions that will be used through the text. In the third section we establish the transformation between the cylindrical and the noncylindrical problems. In the fourth section we define weak solution and finally in the fifth section we prove our fundamental result of existence of weak solution.

2 Preliminaries

Let $\kappa : [0,T] \to I\!R^{n^2}$ be a function such that $\kappa(t)$ is a $n \times n$ matrix. Let Ω be an bounded open set of $I\!R^n$ with a smooth boundary Γ. We can suppose without loss of generality that $0 \in \Omega$. Consider the sets

$$\Omega_t = \{x = \kappa(t)\, y,\ y \in \Omega\},$$

where $\kappa(t) = (\alpha_{ij}(t))_{n\times n}$, $y = (y_{1,\ldots,}y_n) \in \Omega \subset I\!R^n$ and

$$x = \kappa(t)\, y = \left(\sum_{j=1}^{n} \alpha_{1j}(t)\, y_j, ..., \sum_{j=1}^{n} \alpha_{nj}(t)\, y_j \right) \in \Omega_t. \tag{1}$$

We use the notation $\alpha_i \beta_i = \sum_{i=1}^{n} \alpha_i \beta_i$,

$$\kappa(t) = (\alpha_{ij}(t)) \quad \text{and} \quad \kappa^{-1}(t) = (\beta_{ij}(t)), \tag{2}$$

where $\alpha_{ij}(t)$ are C^1 functions defined on $[0,T]$ such that $\det \kappa(t) > 0$. To transform a noncylindrical problem in one defined in a cylindrical domain, we introduce the functions:

$$u(x,t) = v\left(\kappa^{-1}(t)\, x, t\right), \qquad f(x,t) = g\left(\kappa^{-1}(t)\, x, t\right), \tag{3}$$

$$p(x,t) = q\left(\kappa^{-1}(t)\, x, t\right), \qquad u_0(x) = v_0\left(\kappa^{-1}(0)\, x\right), \tag{4}$$

$$\eta(x,t) = N\left(\kappa^{-1}(t)\, x, t\right), \qquad F(\eta(x,t)) = G\left(N\left(\kappa^{-1}(t)\, x, t\right)\right). \tag{5}$$

We introduce the following spaces to obtain the main results:

$$\nu_t = \{\varphi \in (\mathcal{D}(\Omega_t))^n \ ;\ div\ \varphi = 0\} \quad \text{and}$$

$$V_s(\Omega_t) \ \text{ the closure of } \ \nu_t \ \text{ with the norm of } \ (H^s(\Omega_t))^n,\ s \in I\!R_+.$$

In the special cases $s = 0$ and $s = 1$, we use $V(\Omega_t) = V_1(\Omega_t)$ and $H(\Omega_t) = V_0(\Omega_t)$. The inner product of $V(\Omega_t)$, $H(\Omega_t)$ and $(H^s(\Omega_t))^n$are defined by:

$$(u,z)_{H(\Omega_t)} = \int_{\Omega_t} u_i(x)\, z_i(x)\, dx, \qquad ((u,z))_{V(\Omega_t)} = \int_{\Omega_t} \frac{\partial u_i(x)}{\partial x_j} \frac{\partial z_i(x)}{\partial x_j} dx$$

and $((u,z))_s = (u_i, z_i)_{H^s(\Omega_t)}$.
Remark: $\mathrm{V}_s(\Omega_t)$ is continuously imbedded in $(H_0^1(\Omega_t))^n$ for $s \geq n/2$, since $s \geq 1$, $V_s \hookrightarrow V \hookrightarrow H \equiv H' \hookrightarrow V' \hookrightarrow V_s'$.

In a similar way, we have the spaces over Ω:

$$\nu = \left\{\psi \in (\mathcal{D}(\Omega))^n \ ;\ div\left(\kappa^{-1}(t)\psi^t\right) = 0\right\} \quad \text{and}$$

$$V(\Omega) \ \text{ the closure of } \nu \text{ with the norm of } \ (H^s(\Omega))^n,$$

and, for $s = 0$ and $s = 1$, we also define $V = V_1(\Omega)$ and $H = V_0(\Omega)$, provided with the following inner products

$$(v, w)_H = \int_\Omega v_i(x)\, w_i(x)\, dx, \qquad ((v, w))_V = \int_\Omega \frac{\partial v_i(x)}{\partial x_j} \frac{\partial w_i(x)}{\partial x_j} dx,$$

and with the norms $\|v\|_H = (v, v)_H^{1/2}$ and $\|v\|_V = ((v, v))_V^{1/2}$. We introduce now the bilinear and trilinear forms corresponding to the variational formulation to the cylindrical and noncylindrical problems.

In the noncylindrical case we define

$$\hat{a}(t; u, z) = \int_{\Omega_t} \frac{1}{\eta} \frac{\partial u_i(x)}{\partial x_j} \frac{\partial z_i(x)}{\partial x_j} dx, \tag{6}$$

$$\hat{a}_1(t; u, z) = \int_{\Omega_t} \frac{\partial u_i(x)}{\partial x_j} z_i(x) \frac{\partial}{\partial x_j} \left(\frac{1}{\eta}\right) dx, \tag{7}$$

$$\hat{b}(t; u, z, \xi) = \int_{\Omega_t} \frac{u_i(x)}{\eta} \frac{\partial}{\partial x_i} \left(\frac{z_j}{\eta}\right) \xi_j(x)\, dx, \tag{8}$$

$$\hat{d}(t; u, w) = \int_{\Omega_t} F(\eta)\, u_i(x) \frac{w_i(x)}{\eta} dx, \tag{9}$$

$$\hat{e}(t; u, w) = \int_{\Omega_t} u_i(x) \frac{\eta'}{\eta^2} w_i(x)\, dx, \tag{10}$$

and in the cylindrical case

$$a_1(t; v, w) = \int_\Omega \frac{1}{N} \beta_{lj}(t)\, \beta_{rj}(t) \frac{\partial v_i}{\partial y_r} \frac{\partial w_i}{\partial y_l} dy\ , \tag{11}$$

$$a_2(t; v, w) = -\int_\Omega \beta_{lj}(t)\, \beta_{rj}(t) \frac{\partial v_i}{\partial y_r} \frac{\partial N}{\partial y_l} \frac{w_i}{N^2} dy, \tag{12}$$

$$b_1(t; v, w, \psi) = \int_\Omega \frac{v_i}{N^2} \beta_{li}(t) \frac{\partial w_j}{\partial y_l} \psi_j dy\ , \tag{13}$$

$$b_2(t; v, w, \psi) = -\int_\Omega \frac{v_i}{N^3} \beta_{li}(t) \frac{\partial N}{\partial y_l} w_j \psi_j dy, \tag{14}$$

$$c(t; v, w) = \int_\Omega \beta'_{lr}(t)\, \alpha_{rj}(t)\, y_j \frac{\partial v_i(y)}{\partial y_l} \frac{w_i(y)}{N} dy, \tag{15}$$

$$d(t;v,w) = \int_{\Omega} G(N)\, v_i(y)\, \frac{w_i(y)}{N} dy, \tag{16}$$

$$e(t;v,w) = \int_{\Omega} v_i(y)\, \frac{N'}{N^2} w_i(y)\, dy. \tag{17}$$

Lemma 1 *If $\kappa(t)$ satisfies the hypotheses as stated before, then*

$$\frac{(\det \kappa(t))'}{\det \kappa(t)} = -\mathrm{tr}((\kappa^{-1}(t))'\kappa(t)).$$

3 The Equation in Q

If $x \in \Omega_t$ and $y \in \Omega$ satisfy (1), using (2) we have

$$x_r = \alpha_{rj}(t)\, y_j \quad e \quad y_l = \beta_{lr}(t)\, x_r, \tag{18}$$

$$\frac{\partial y_l}{\partial t} = \beta'_{lr}(t)\, x_r = \beta'_{lr}(t)\, \alpha_{rj}(t)\, y_j, \tag{19}$$

$$\frac{\partial y_l}{\partial x_j} = \beta_{lj}(t) \ . \tag{20}$$

From (3), $u_i(x,t) = v_i(y,t)$, we get

$$\frac{\partial u_i(x,t)}{\partial x_j} = \frac{\partial v_i(y,t)}{\partial y_l} \frac{\partial y_l}{\partial x_j}, \tag{21}$$

and using (20) we obtain

$$\frac{\partial u_i(x,t)}{\partial x_j} = \beta_{lj}(t)\, \frac{\partial v_i(y,t)}{\partial y_l}, \tag{22}$$

For j fixed, we also have

$$\frac{\partial^2 u_i(x,t)}{\partial x_j^2} = \frac{\partial}{\partial x_j}\left(\beta_{lj}(t)\, \frac{\partial v_i(y,t)}{\partial y_l}\right) = \beta_{lj}(t)\, \frac{\partial}{\partial x_j}\left(\frac{\partial v_i(y,t)}{\partial y_l}\right),$$

$$\frac{\partial^2 u_i(x,t)}{\partial x_j^2} = \beta_{lj}(t)\, \frac{\partial^2 v_i(y,t)}{\partial y_r \partial y_l} \frac{\partial y_r}{\partial x_j},$$

so

$$\Delta u_i(x,t) = \beta_{lj}(t)\, \beta_{rj}(t)\, \frac{\partial^2 v_i(y,t)}{\partial y_r \partial y_l}. \tag{23}$$

On the other hand,

$$u_i(x,t)\frac{\partial}{\partial x_i}\left(\frac{u}{\eta}\right)(x,t) = \frac{u_i(x,t)}{\eta}\frac{\partial u}{\partial x_i}(x,t) + u_i(x,t)\frac{\partial}{\partial x_i}\left(\frac{1}{\eta}\right)u(x,t) =$$
$$= \frac{u_i(x,t)}{\eta}\frac{\partial u}{\partial x_i}(x,t) - \frac{u_i(x,t)}{\eta^2}\frac{\partial \eta}{\partial x_i}u(x,t)$$

Then

$$u_i(x,t)\frac{\partial}{\partial x_i}\left(\frac{u}{\eta}\right)(x,t) = \frac{u_i(x,t)}{\eta}\frac{\partial u_j}{\partial x_i}(x,t) - \frac{u_i(x,t)}{\eta^2}\frac{\partial \eta}{\partial x_i}u_j(x,t)$$

for $j = 1, 2, ..., n$ and using (22)

$$u_i(x,t)\frac{\partial}{\partial x_i}\left(\frac{u}{\eta}\right)(x,t) = \frac{v_i(y,t)}{N}\beta_{li}(t)\frac{\partial v_j(y,t)}{\partial y_l} - \frac{v_j(y,t)}{N^2}\beta_{li}(t)\frac{\partial N}{\partial y_l}v_j(y,t). \tag{24}$$

Since $\frac{\partial u_i(x,t)}{\partial t} = \frac{\partial v_i(y,t)}{\partial y_l}\frac{\partial y_l}{\partial t} + \frac{\partial v_i(y,t)}{\partial t}$, using (19) we get

$$\frac{\partial u_i(x,t)}{\partial t} = \beta'_{lr}(t)\,\alpha_{rj}(t)\,y_j\frac{\partial v_i(y,t)}{\partial y_l} + \frac{\partial v_i(y,t)}{\partial t}. \tag{25}$$

From (4) we have $\frac{\partial p(x,t)}{\partial x_i} = \frac{\partial q(y,t)}{\partial y_l}\frac{\partial y_l}{\partial x_i} = \beta_{li}(t)\frac{\partial q(y,t)}{\partial y_l}$, hence and by (20),

$$\frac{\partial p(x,t)}{\partial x_i} = \left(\frac{\partial q(y,t)}{\partial y_1}, ..., \frac{\partial q(y,t)}{\partial y_n}\right)(\beta_{1i}(t), ..., \beta_{ni}(t)\ ) = \left(\nabla q.\kappa^{-1}(t)\right)_i,$$

that is
$$\nabla p(x,t) = \nabla q(y,t).\kappa^{-1}(t). \tag{26}$$

For each i fixed, from (22), the following relations hold:

$$\frac{\partial u_i(x,t)}{\partial x_i} = \sum_{l=1}^{n}\beta_{li}(t)\frac{\partial v_i(y,t)}{\partial y_l} \text{ and } div\ u(x,t) = \sum_i \frac{\partial u_i(x,t)}{\partial x_i}$$

Therefore $div\ u(x,t) = \beta_{li}(t)\frac{\partial v_i(y,t)}{\partial y_l}$. (27)

So

$$div\ u(x,t) = \left[\frac{\partial}{\partial y_1}(\beta_{1i}v_i(y,t)) + ... + \frac{\partial}{\partial y_n}(\beta_{ni}v_i(y,t))\right],$$

$$div\ u(x,t) = \left[\left(\frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_n}\right)\left(\kappa^{-1}(t).v^t(y,t)\right)\right] = \nabla.\left(\kappa^{-1}(t).v^t(y,t)\right),$$

therefore

$$div\ u(x,t) = div\left(\kappa^{-1}(t).v^t(y,t)\right). \tag{28}$$

From (5) $\dfrac{\partial \eta}{\partial x_j} = \dfrac{\partial N}{\partial y_l}\dfrac{\partial y_l}{\partial x_j}$, by (20), we have

$$\frac{\partial \eta}{\partial x_j} = \beta_{lj}(t)\frac{\partial N}{\partial y_l}. \tag{29}$$

On the other hand, $\dfrac{\partial \eta}{\partial t} = \dfrac{\partial N}{\partial y_l}\dfrac{\partial y_l}{\partial t} + \dfrac{\partial N}{\partial t}$, using (19) we get

$$\frac{\partial \eta}{\partial t} = \beta'_{lt}(t)\,\alpha_{rj}(t)\,y_j\frac{\partial N}{\partial y_l} + \frac{\partial N}{\partial t}. \tag{30}$$

Replacing (23) to (30) in (PNC), we conclude for $i = 1, ..., n$

$$\begin{aligned}
&\beta'_{lr}(t)\,\alpha_{rj}(t)\,y_j\frac{\partial v_i(y,t)}{\partial y_l} + \frac{\partial v_i(y,t)}{\partial t} - \mu\beta_{lj}(t)\,\beta_{rj}(t)\frac{\partial^2 v_i(y,t)}{\partial y_r \partial y_l} + \\
&\quad + \frac{v_j(y,t)}{N}\beta_{lj}(t)\frac{\partial v_i(y,t)}{\partial y_l} - \frac{v_j(y,t)}{N^2}\beta_{lj}(t)\frac{\partial N}{\partial y_l}v_i(y,t) + \\
&+ N(y,t)\ \frac{\partial q(y,t).\kappa^{-1}(t)}{\partial y_l} + G(N(y,t))\,v_i(y,t) = N(y,t)\,g_i(y,t) \ \text{ in } \ Q\ .
\end{aligned}$$

So

$$\begin{cases}
v' - \mu\dfrac{\partial}{\partial y_l}\left(\beta_{lj}(t)\,\beta_{rj}(t)\dfrac{\partial v(y,t)}{\partial y_r}\right) + \dfrac{v_j(y,t)}{N}\beta_{lj}(t)\dfrac{\partial v(y,t)}{\partial y_l} \\
-\dfrac{v_j(y,t)}{N^2}\beta_{lj}(t)\dfrac{\partial N}{\partial y_l}v(y,t) + \beta'_{lr}(t)\,\alpha_{rj}(t)\,y_j\dfrac{\partial v(y,t)}{\partial y_l} + \\
N(y,t)\,\nabla q(y,t).\kappa^{-1}(t) + \mu G(N(y,t))\,v(y,t) = N(y,t)\,g(y,t) \quad \text{in } \ Q \\
\\
div\,[\kappa^{-1}(t).v^t(y,t)] = 0 \ \text{ in } Q \\
v = 0 \text{ in } \Sigma \\
v(y,0) = v_0(y) \ \text{ with } y \in \Omega.
\end{cases} \tag{PC}$$

4 Definition of Weak Solution

Definition 1 *Notion of weak solution to the problem (PNC).*

To define the weak solution of (PNC) we will eliminate the pressure as in classical Navier-Stokes equations. For that, we make the inner product in $L^2(\Omega)$ of the equation (divided by η) by a function of V and observing that

$$\frac{d}{dt}\left(\frac{u}{\eta}\right) = \frac{u'}{\eta} - \frac{u\eta'}{\eta^2}$$

we can rewrite (PNC) in the following way:

For f, η and u_0 given, with $f \in L^2\left(0,T,V\left(\Omega_t\right)'\right)$ and $u_0 \in H\left(\Omega_t\right)$ we have to find u satisfying $u \in L^2\left(0,T,V\left(\Omega_t\right)\right)$ and:

$$\begin{cases} -\left(u,\dfrac{\xi'}{\eta}\right)+\left(\dfrac{\eta'}{\eta^2}u,\xi\right)+\left(\dfrac{u}{\eta}\cdot\nabla\left(\dfrac{u}{\eta}\right),\xi\right)+\mu\left(\nabla\left(u\right),\nabla\left(\dfrac{1}{\eta}\right)\xi\right) \\ +\mu\left(\nabla\left(u\right),\nabla\left(\xi\right)\frac{1}{\eta}\right)+\mu\left(\dfrac{F(\eta)}{\eta}u,\xi\right)=\langle f,\xi\rangle \quad \forall\xi\in V \text{ in } \mathcal{D}'\left(0,T\right) \\ u(0)=u_0\in H. \end{cases} \tag{31}$$

Integrating in $[0,T]$, and making use of (6) and (10), we obtain

$$\begin{cases} u\in L^2\left(0,T,V\left(\Omega_t\right)\right)\cap L^\infty\left(0,T,H\left(\Omega_t\right)\right) \\ -\int_0^T\left(u,\xi'/N\right)_{H(\Omega_t)}dt+\mu\int_0^T\hat{a}\left(t;u,\xi\right)dt+\mu\int_0^T\hat{a}_1\left(t;u,\xi\right)dt+ \\ \int_0^T\hat{b}\left(t;u,u,\xi\right)dt+\int_0^T\hat{d}\left(t;u,\xi\right)dt+\int_0^T\hat{e}\left(t;u,\xi\right)dt=\int_0^T\left(f,\xi\right)_{H(\Omega_t)}dt \\ \forall\xi\in L^2\left(0,T;V\left(\Omega_t\right)\cap L^n\left(\Omega_t\right)^n\right),\ \xi'\in L^2\left(0,T;H\left(\Omega_t\right)\right) \\ \xi\left(0\right)=0,\xi\left(T\right)=0 \\ u\left(0\right)=u_0. \end{cases}$$

(PNC1)

Definition 2 *Notion of weak solution to the problem (PC).*

In this case we divide (PC) by N, then multiply by a suitable function ψ and integrate in Ω, obtaining

$$\begin{aligned} &\int_\Omega\frac{v'}{N}\psi dy-\int_\Omega\frac{\mu}{N}\frac{\partial}{\partial y_l}\left(\beta_{lj}\left(t\right)\beta_{rj}\left(t\right)\frac{\partial v}{\partial y_l}\right)\psi dy+\int_\Omega\frac{v_j\left(y,t\right)}{N^2}\beta_{lj}\left(t\right)\frac{\partial v\left(y,t\right)}{\partial y_l}\psi dy \\ &\quad-\int_\Omega\frac{v_j\left(y,t\right)}{N^3}\beta_{lj}\left(t\right)\frac{\partial N}{\partial y_l}v\left(y,t\right)\psi dy+\int_\Omega\frac{1}{N}\,\beta'_{lr}\left(t\right)\alpha_{rj}\left(t\right)y_j\frac{\partial v}{\partial y_l}\psi dy+ \\ &\quad+\int_\Omega\nabla q\left(y,t\right).\kappa^{-1}\left(t\right)\psi dy+\int_\Omega\frac{\mu G\left(N\right)}{N}v\psi dy=\int_\Omega g\ \psi dy. \end{aligned}$$

Making use of (11) to (17), we have

$$\int_\Omega\frac{v'}{N}.\psi dy=\int_\Omega\left(\left(\frac{v'}{N}-\frac{vN'}{N^2}\right).\psi+\frac{vN'}{N^2}\psi\right)dy=-\left(\frac{v}{N},\psi'\right)+e\left(t;v,\psi\right)$$

$$\begin{aligned} &\int_\Omega\frac{1}{N}\frac{\partial}{\partial y_l}\left(\beta_{lj}\left(t\right)\beta_{rj}\left(t\right)\frac{\partial v}{\partial y_l}\right)\psi dy=\int_\Omega\beta_{lj}\left(t\right)\beta_{rj}\left(t\right)\frac{\partial v_i}{\partial y_r}\frac{\partial}{\partial y_l}\left(\frac{\psi_i}{N}\right)dy= \\ &\int_\Omega\frac{1}{N}\beta_{lj}\left(t\right)\beta_{rj}\left(t\right)\frac{\partial v_i}{\partial y_r}\frac{\partial\psi_i}{\partial y_l}dy-\int_\Omega\beta_{lj}\left(t\right)\beta_{rj}\left(t\right)\frac{\partial v_i}{\partial y_r}\frac{\partial N}{\partial y_l}\frac{\psi_i}{N^2}dy= \\ &=a_1\left(t;v,\psi\right)+a_2\left(t;v,\psi\right) \end{aligned}$$

$$\int_\Omega \frac{1}{N}\beta_{li}(t)\, v_i \frac{\partial}{\partial y_l}\left(\frac{v}{N}\right)\psi dy = \int_\Omega \beta_{li}(t)\, v_i \frac{\partial v_j}{\partial y_l}\frac{\psi_j}{N^2}dy -$$
$$\int_\Omega \beta_{li}(t)\,\frac{v_i}{N^3}\frac{\partial N}{\partial y_l}v_j\psi_j dy = b_1(t;v,v,\psi) + b_2(t;v,v,\psi)$$

$$\int_\Omega \frac{1}{N}\ \beta'_{lr}(t)\,\alpha_{rj}(t)\, y_j \frac{\partial v}{\partial y_l}\psi dy = \int_\Omega \beta'_{lr}(t)\,\alpha_{rj}(t)\, y_j \frac{\partial v_i}{\partial y_l}\frac{\psi_i}{N}dy = c(t;v,\psi)$$

$$\int_\Omega \nabla q \kappa^{-1}(t).\psi dy = \int_\Omega q.div\left(\kappa^{-1}(t)\psi^t\right)dy = 0.$$

$$\int_\Omega \frac{G(N)}{N}v\psi dy = \int_\Omega G(N)\, v_i \frac{\psi_i}{N}dy = d(t;v,\psi) \text{ and } \int_\Omega g.\psi dy = \int_\Omega g_i.\psi_i dy = (g,\psi)$$

so

$$\begin{cases} v \in L^2(0,T;V)\cap L^\infty(0,T;H) \\ -\int_0^T \left(\frac{v}{N},\psi'\right)dt + \mu\int_0^T a(t;v,\psi)\,dt + \int_0^T b_1(t;v,v,\psi)\,dt + \int_0^T b_2(t;v,v,\psi)\,dt \\ +\int_0^T c(t;v,\psi)\,dt + \mu\int_0^T d(t;v,\psi)\,dt + \int_0^T e(t;v,\psi)\,dt = \int_0^T (g,\psi)\,dt \\ \forall\psi \in L^2(0,T;V\cap L^n(\Omega)^n),\ \psi' \in L^2(0,T;H) \\ \psi(0)=0, \psi(T)=0, \quad v(0)=v_0 \end{cases} \tag{PC1}$$

Theorem 1 *Problems (PNC1) and (PC1) are equivalent.*

Proof. Recalling from (1),(2) and (3) we have

$$x = \kappa(t)\, y, \qquad y = \kappa^{-1}(t)\, x, \quad x_r = \alpha_{rj}y_j, \qquad y_l = \beta_{lr}x_r\ ,$$

we have established that $u(x,t) = v\left(\kappa^{-1}(t)\, x, t\right)$.
Let $\xi(x,t) = \left|\det \kappa^{-1}(t)\right| \psi\left(\kappa^{-1}(t)\, x, t\right)$. Then

$$\frac{\partial \xi_i(x,t)}{\partial t} = \left|\det \kappa^{-1}(t)\right|\left(\frac{\partial \psi_i(y,t)}{\partial y_l}\frac{\partial y_l}{\partial t} + \frac{\partial \psi_i(y,t)}{\partial t}\right) + \left|\det \kappa^{-1}(t)\right|'\psi_i(y,t),$$

$$\frac{\partial \xi_i(x,t)}{\partial t} = \left|\det \kappa^{-1}(t)\right|\left(\frac{\partial \psi_i}{\partial y_l}\beta'_{lr}\alpha_{rj}y_j + \frac{\partial \psi_i}{\partial t}\right) + \left|\det \kappa^{-1}(t)\right|'\psi_i, \tag{32}$$

Now we compute each integral in Ω_t of problem (PNC1) and making use of (32) we have

$$\left(u, \frac{\xi^{'}}{\eta}\right)_{H(\Omega_t)} = \int_{\Omega} \left(\frac{v_i}{N}\frac{\partial \psi_i}{\partial y_l}\beta'_{lr}\alpha_{rj}y_j + \frac{v_i}{N}\frac{\partial \psi_i}{\partial t} - \frac{1}{N}\frac{(\det \kappa(t))'}{\det \kappa(t)}v_i\psi_i\right) dy \left|\det \kappa^{-1}(t)\right| \tag{33}$$

On the other hand $\beta'_{lr}\alpha_{rl} = t_r\left(\left(\kappa^{-1}(t)\right)'\kappa(t)\right)$ then, by lemma 1

$$\beta'_{lr}\alpha_{rl} = -\frac{(\det \kappa(t))'}{\det \kappa(t)}, \tag{34}$$

where $t_r(A)$ denotes the trace of the $n \times n$ matrix A.

Let F be the following vector field $\left(0, ..., \frac{v_i\psi_i}{N}\beta'_{lr}\alpha_{rj}y_j, 0, ..., 0\right)$ where the non null component ocupies the $l-$position , then

$$\begin{aligned} divF &= \frac{\partial}{\partial y_l}\left(\frac{v_i\psi_i}{N}\left[\beta'_{lr}\alpha_{rj}y_j\right]\right) = \frac{\partial}{\partial y_l}\left(\frac{v_i\psi_i}{N}\right).\left[\beta'_{lr}\alpha_{rj}y_j\right] + \frac{v_i\psi_i}{N}\frac{\partial}{\partial y_l}\left(\beta'_{lr}\alpha_{rj}y_j\right) = \\ &= \frac{1}{N}\frac{\partial v_i}{\partial y_l}\psi_i\beta'_{lr}\alpha_{rj}y_j + v_i\frac{\partial}{\partial y_l}\left(\frac{\psi_i}{N}\right)\beta'_{lr}\alpha_{rj}y_j + \frac{v_i\psi_i}{N}\beta'_{lr}\frac{\partial}{\partial y_l}\left(\alpha_{rj}y_j\right), \end{aligned}$$

but

$$\frac{\partial}{\partial y_l}\left(\alpha_{rj}y_j\right) = \frac{\partial}{\partial y_l}\left(\alpha_{r1}y_1 + ... + \alpha_{rn}y_n\right) = \alpha_{rl},$$

then

$$divF = \frac{1}{N}\frac{\partial v_i}{\partial y_l}\psi_i\beta'_{lr}\alpha_{rj}y_j + v_i\frac{\partial}{\partial y_l}\left(\frac{\psi_i}{N}\right)\beta'_{lr}\alpha_{rj}y_j + \frac{v_i\psi_i}{N}\beta'_{lr}\alpha_{rl},$$

and making use of (34)

$$divF = \frac{1}{N}\frac{\partial v_i}{\partial y_l}\psi_i\beta'_{lr}\alpha_{rj}y_j + v_i\frac{\partial}{\partial y_l}\left(\frac{\psi_i}{N}\right)\beta'_{lr}\alpha_{rj}y_j - \frac{1}{N}\frac{(\det \kappa(t))'}{\det \kappa(t)}v_i\psi_i.$$

From Green's Theorem, $\int_{\Omega} div\ F dy = \int_{\hat{\Sigma}} F.\vec{\eta}\ ds = 0$, then

$$\int_{\Omega}\left(\frac{1}{N}\frac{\partial v_i}{\partial y_l}\psi_i\beta'_{lr}\alpha_{rj}y_j + v_i\frac{\partial}{\partial y_l}\left(\frac{\psi_i}{N}\right)\beta'_{lr}\alpha_{rj}y_j - \frac{1}{N}\frac{(\det \kappa(t))'}{\det \kappa(t)}v_i\psi_i\right)dy = 0,$$

and

$$\begin{aligned} &-\int_{\Omega}\left(\frac{v_i}{N}\frac{\partial \psi_i}{\partial y_l}\beta'_{lr}\alpha_{rj}y_j - \frac{1}{N}\frac{(\det \kappa(t))'}{\det \kappa(t)}v_i\psi_i\right)dy = \\ &= \int_{\Omega}\left(v_i\psi_i\frac{\partial}{\partial y_l}\left(\frac{1}{N}\right)\beta'_{lr}\alpha_{rj}y_j + \frac{\beta'_{lr}\alpha_{rj}}{N}y_j\frac{\partial v_i}{\partial y_l}\psi_i\right)dy, \end{aligned} \tag{35}$$

combining (35) and (33), we get

$$\left(u, \frac{\xi^{'}}{\eta}\right)_{H(\Omega_t)} = \int\limits_{\Omega} \frac{v_i}{N}\frac{\partial \psi_i}{\partial t} - \frac{\beta'_{lr}\alpha_{rj}}{N} y_j \frac{\partial v_i}{\partial y_l}\psi_i - v_i\psi_i \frac{\partial}{\partial y_l}\left(\frac{1}{N}\right)\beta'_{lr}\alpha_{rj}y_j dy \left|\det \kappa^{-1}(t)\right|,$$

$$-\left(u, \frac{\xi^{'}}{\eta}\right)_{H(\Omega_t)} = \left[-\left(v, \frac{\psi'}{N}\right) + c(t; v, \psi) + c_1(t; v, \psi)\right]\left|\det \kappa^{-1}(t)\right|, \quad (36)$$

where

$$\int\limits_{\Omega} v_i\psi_i \frac{\partial}{\partial y_l}\left(\frac{1}{N}\right)\beta'_{lr}\alpha_{rj}y_j dy \left|\det \kappa^{-1}(t)\right| = c_1(t; v, \psi).$$

From (6) and making use of (22) we have

$$\hat{a}(t; u, \xi) = \left[\int\limits_{\Omega} \frac{\beta_{lj}\beta_{rj}}{N}\frac{\partial v_i}{\partial y_l}\frac{\partial \psi_i}{\partial y_r} dy\right]\left|\det \kappa^{-1}(t)\right| = a_1(t; v, \psi)\left|\det \kappa^{-1}(t)\right|, \quad (37)$$

but

$$\hat{a}_1(t; u, \xi) = \int\limits_{\Omega_t} \frac{\partial u_i(x)}{\partial x_j}\xi_i(x)\frac{\partial}{\partial x_j}\left(\frac{1}{\eta}\right)dx =$$

$$= -\int\limits_{\Omega} \frac{1}{N^2}\beta_{lj}(t)\beta_{rj}(t)\frac{\partial v_i(y)}{\partial y_l}\frac{\partial N}{\partial y_r}\psi_i(y)\, dy \left|\det \kappa^{-1}(t)\right| = a_2(t; v, \psi)\left|\det \kappa^{-1}(t)\right|. \quad (38)$$

From (8), $\hat{b}(t; u, u, \xi) =$

$$= \left[\int\limits_{\Omega} \frac{v_i}{N^2}\beta_{lj}(t)\frac{\partial v_j}{\partial y_l}\psi_j(y)\, dy - \int\limits_{\Omega} \frac{v_i}{N^3}\beta_{lj}(t)\, v_j \frac{\partial N}{\partial y_l}\psi_j(y)\, dy\right]\left|\det \kappa^{-1}(t)\right|$$

then by (13) and (14), we have

$$\hat{b}(t; u, u, \xi) = \left(b_1(t; v, v, \psi) + b_2(t; v, v, \psi)\right)\left|\det \kappa^{-1}(t)\right|. \quad (39)$$

From (9)

$$\hat{d}(t; u, \xi) = \int\limits_{\Omega} G(N)\, v_i(y)\frac{\psi_i(y)}{N} dy \left|\det \kappa^{-1}(t)\right| = d(t; v; \psi)\left|\det \kappa^{-1}(t)\right|. \quad (40)$$

We also have from (10)

$$\hat{e}(t; u, \xi) = \left[\int_{\Omega} \frac{v_i(y)}{N^2}\beta'_{lr}(t)\,\alpha_{rj}y_j\frac{\partial N}{\partial y_l}\psi_i(y)\, dy + \int_{\Omega} \frac{v_i(y)}{N^2}\frac{\partial N}{\partial t}\psi_i(y)\, dy\right]\left|\det \kappa^{-1}(t)\right|$$
$$= \left[-c_1(t; v, \psi) + e(t; v, \psi)\right]\left|\det \kappa^{-1}(t)\right| \quad (41)$$

Finally, integrating both sides of expressions (36) to (41) and adding term by term we conclude that problems (PNC1) and (PC1) are equivalent.

□

Define the forms concerning to the cylindrical problem:

$$a\left(t;v,w\right)=\ a_1\left(t;v,w\right)+\ a_2\left(t;v,w\right)=\int_{\Omega}\beta_{lj}\left(t\right)\beta_{rj}\left(t\right)\frac{\partial vi}{\partial y_r}\frac{\partial}{\partial y_l}\left(\frac{w_i}{N}\right)dy\ . \quad (42)$$

$$b\left(t;v,w,\psi\right)=b_1\left(t;v,w,\psi\right)+\ b_2\left(t;v,w,\psi\right)=\int_{\Omega}\beta_{li}\left(t\right)v_i\frac{\partial}{\partial y_l}\left(\frac{w_j}{N}\right)\frac{\psi_j}{N}dy\ . \quad (43)$$

Definition 3 *Let* $A\left(t\right):\left(H_0^1\left(\Omega\right)\right)^n\rightarrow\left(H^{-1}\left(\Omega\right)\right)^n$ *be the operator defined by*

$$A\left(t\right)v=-\frac{1}{N}\frac{\partial}{\partial y_l}\left(\beta_{lj}\left(t\right)\beta_{rj}\left(t\right)\frac{\partial v\left(y,t\right)}{\partial y_r}\right)\ \ for\quad v\in\left(H_0^1\left(\Omega\right)\right)^n\ , \quad (44)$$

Lemma 2 *The linear form* $a\left(t;v,w\right)$ *defined in (42) and the operator* $A\left(t\right)$ *defined in (44) satisfy*

$$\begin{array}{ll} i. & \left\langle A\left(t\right)v,\ w\right\rangle=a\left(t;v,w\right)\ ,\quad \forall v,w\in V \\ ii. & \left|a\left(t;v,w\right)\right|\leq C\left\|v\right\|\left\|w\right\|\ ,\quad \forall v,w\in V. \end{array}$$

Lemma 3 *The linear form* $a_1\left(t;v,w\right)$ *as defined in (11) is coercive and continuous, that is,*

i. $a_1\left(t;v,v\right)\geq a_0\left\|v\right\|^2$, $\forall v\in V$, *where* $a_0>0$

ii. $a_1\left(t;v,w\right)\leq C\left\|v\right\|\left\|w\right\|$, $\forall v$,$w\in V$.

Lemma 4 *Let* $b\left(t;v,w,\psi\right)$, $c\left(t;v,w\right)$, $d\left(t;v,w\right)$ *and* $e\left(t;v,w\right)$ *the multilinear forms defined by (43) , (15), (16) and (17) respectively. Then*

i. $\left|b\left(t;v,w,\psi\right)\right|\leq C\left\|v\right\|\left\|w\right\|\left\|\psi\right\|_{V\cap\left(L^n\left(\Omega\right)\right)^n}\quad\forall v,w\in V$ *and* $\psi\in V\cap\left(L^n\left(\Omega\right)\right)^n$.

ii. $b\left(t;v,v,w\right)=-b\left(t;v,w,v\right)\ \ \forall v\in V$ *and* $w\in V_s\left(\Omega\right)$, *where* $s=\dfrac{n}{2}$.

iii. For $v\in V$, *the linear form* $w\mapsto b\left(t;v,v,w\right)$ *is continuous on* $V_s\left(\Omega\right)$ $\left(s=\dfrac{n}{2}\right)$ *and* $b\left(t;v,v,w\right)=\left\langle B\left(t\right)v,\ w\right\rangle_{V_s'V_s}$, *where* $B\left(t\right)v\in V_s'\left(\Omega\right)$ *and*

$$\left\|B\left(t\right)v\right\|_{V_s'}\leq C\left\|v\right\|^2_{\left(L^p\left(\Omega\right)\right)}\qquad with\ \frac{1}{p}=\frac{1}{2}-\frac{1}{2n}. \quad (45)$$

iv. $\left|c\left(t;v,w\right)\right|\leq C\left\|v\right\|\left\|w\right\|\quad\forall v,w\in H$.

v. For $v\in V$, *the linear form* $w\mapsto c\left(t;v,w\right)$ *is continuous on* H *and*

$$c\left(t;v,w\right)=\ \left\langle C\left(t\right)v,w\right\rangle_{H'H}=\left(C\left(t\right)v,w\right),$$

where $C(t)\,v \in H' = H$ *and* $|C(t)\,v| \leq C\,\|v\|$.
vi. For $v \in V$, *the linear form* $w \mapsto d(t;v,w)$ *is continuous on* H,

$$d(t;v,w) = \langle D(t)\,v, w\rangle_{H'H} = (D(t)\,v, w),$$

and $|D(t)\,v| \leq C\,\|v\|$.
vii. For $v \in V$, *the linear form* $w \mapsto e(t;v,w)$ *is continuous on* H,

$$e(t;v,w) = \langle E(t)\,v, w\rangle_{H'H} = (E(t)\,v, w)$$

and $|E(t)\,v| \leq C\,\|v\|$.
viii. $L^2(0,T;V) \cap L^\infty(0,T;H) \subset L^4(0,T;(L^p(\Omega))^n)$, *where* p *is given by (45).*

The three last lemmas can be found in MIRANDA [7], pg 253-254.

5 Existence of Solution

In this section we will prove the following result

Theorem 2 *(existence of weak solutions)*

Let $\Omega \subset \mathbb{R}^n$,with $n = 2, 3$, be an open bounded set with regular boundary and $T > 0$. Also, let be given, $u_0 \in H(\Omega_T)$, $g \in L^2(0,T,V_s'(\Omega_T))$ and a continuous function $F : (0,1] \to \mathbb{R}^+$. Suppose that the porosity $n : \hat{Q} \to (0,1]$ satisfies

$$0 < n_0 \leq n(x,t) < 1 \quad \forall (x,t) \in \hat{Q}$$
$$n' \in L^2\left(0,T,L^{\frac{3}{2}}(\Omega_T)\right) \cap L^1(0,T,L^\infty(\Omega_T))$$
$$\nabla n \in L^2(0,T,L^\infty(\Omega_T)) \cap L^\infty(0,T,L^3(\Omega_T))$$

Then, there exists a solution $u \in L^2(0,T,V(\Omega_T)) \cap L^\infty(0,T,H(\Omega_T))$ of (PNC1).

Proof. For $s = n/2$ the injection $V_s \hookrightarrow H$ is compact, since $V_s \subset V_1 \equiv V \hookrightarrow H$ and $H_0^1(\Omega)$ is compactly imbedded in $L^2(\Omega)$ (see Lions[6], pg 66).

This result guarantees the existence of solution to the spectral problem

$$((w,v))_{H^s(\Omega)} = \lambda(w,v) \quad \forall v \in V_s(\Omega). \tag{46}$$

Consider an orthonormal basis of $V_s(\Omega)$ generated by a countable set of eigenvectors (w_n) corresponding to the set of positive eigenvalues (λ_μ). We will use (w_n) in Galerkin's methods. For each m we define an approximate solution v_m

to the problem (PC1). Let V_m be the subspaces generated by the first vectors $w_1, ..., w_m$, that is, if $v_m(t) \in V_m$, then

$$v_m(t,y) = \sum_{j=1}^{m} h_{jm}(t)\, w_j(y),$$

where h_{jm} are scalar functions defined in $[0,T]$. Consider the approximate problem

$$\begin{cases} \left(\dfrac{v_m'(t)}{N}, w_j\right) + \mu a(t; v_m(t), w_j) + b(t; v_m(t), v_m(t), w_j) + \ c(t; v_m(t), w_j) \\ +\mu d(t; v_m(t), w_j) = (g, w_j) \quad j = 1, ..., m \text{ and } v_m(0) = v_{0m},\ v_{0m} \to v_0 \text{ in } H. \end{cases} \tag{47}$$

Observe that (47) is a system of nonlinear differential equations where h_{jm} are the unknowns. In fact, we have

$$\begin{cases} \left(\dfrac{w_i}{\sqrt{N}}, \dfrac{w_j}{\sqrt{N}}\right) h_{im}'(t) + (\mu a(t; w_i, w_j) + \mu d(t; w_i, w_j) + c(t; w_i, w_j))\, h_{im}(t) \\ +b(t; w_i, w_l, w_j)\, h_{im}(t)\, h_{lm}(t) = (g, w_j) \quad j = 1, ..., m \text{ and } h_{jm}(0)\, w_j = \ v_{0m} \end{cases}$$

Since w_i are linearly independent, the matrix with the entries given by $\left(\dfrac{w_i}{\sqrt{N}}, \dfrac{w_j}{\sqrt{N}}\right)_{1 \le i,j \le n}$ is nonsingular. Then we can use the inverse of this matrix to obtain the nonlinear system

$$\begin{cases} h_{jm}'(t) = \theta_{ji}(t) \ - (\alpha_{ji}(t) + \beta_{ji}(t) + \delta_{ji}(t))\, h_{im}(t) \ - \gamma_{kik}(t)\, h_{im}(t)\, h_{km}(t) \\ h_{jm}(0) = ith \text{ component } v_{0m} \qquad j = 1, ..., m \end{cases}$$

where $\alpha_{ji}(t),\ \beta_{ji}(t),\ \delta_{ji}(t),\ \gamma_{kik}(t) \in I\!R$. From Caratheodory's theorem (see Hale [4] pg 28), this system has a maximal solution defined in some interval $[0, t_m)$. If $t_m < T$, then $|h_{jm}|$ diverge to $+\infty$ as $t \to t_m$, but this cannot happen due to the first estimates that will be proved further. So $t_m = T$. Since the applications $t \to (g(t), w_j)$ belong to $L^2(0,T;H)$, the same result holds to the functions h_{jm}, hence

$$v_m \in L^2(0,T;V) \ \ and\ v_m' \in L^2(0,T;V), \tag{48}$$

for any m. Now we will show the first estimates, that not depend on m, to the functions v_m. After that we will make the limit.

First Estimates

If we multiply (47) by $h_{jm}(t)$ and sum for $j = 1, ..., m$, we have

$$\left(\frac{v_m'}{N}, v_m \right) + \mu a(t; v_m, v_m) + b(t; v_m, v_m, v_m) + c(t; v_m, v_m) + \mu d(t; v_m, v_m) = (g, v_m),$$

From part (ii) of Lemma 4, $b(t; v_m, v_m, v_m) = 0$ and, by (48)

$$\left(\frac{v_m'}{N}, v_m \right) = \frac{1}{2} \frac{d}{dt} \left| \frac{v_m(t)}{\sqrt{N}} \right|^2 + \frac{1}{2} e(t; v_m, v_m),$$

so

$$\frac{1}{2} \frac{d}{dt} \left| \frac{v_m(t)}{\sqrt{N}} \right|^2 + \frac{1}{2} e(t; v_m, v_m) + \mu a_1(t; v_m, v_m) + \mu a_2(t; v_m, v_m) + c(t; v_m, v_m) + \mu d(t; v_m, v_m) = (g, v_m),$$

hence

$$\frac{1}{2} \frac{d}{dt} \left| \frac{v_m(t)}{\sqrt{N}} \right|^2 + \mu a_1(t; v_m, v_m) + \mu d(t; v_m, v_m) = (g, v_m) - \left[\frac{1}{2} e(t; v_m, v_m) + \mu a_2(t; v_m, v_m) + c(t; v_m, v_m) \right]. \tag{49}$$

Now observe that, from part (ii) of lemma 3,

$$a_0 \|v_m\|^2 \leq a_1(t; v_m, v_m),$$

from part (iv) of lemma 4 and Young's inequality,

$$|c(t; v_m, v_m)| \leq C \|v_m\| \left| \frac{v_m}{\sqrt{N}} \right| \leq \varepsilon \|v_m\|^2 + C_\varepsilon \left| \frac{v_m}{\sqrt{N}} \right|^2,$$

$$\langle g, v_m \rangle \leq \|g\|_{H^{-1}} \|v_m\| \leq C_\varepsilon \|g\|_{H^{-1}}^2 + \varepsilon \|v_m\|^2,$$

$$|a_2(t; v_m, v_m)| \leq \int_\Omega |\beta_{lj}(t)| \, |\beta_{rj}(t)| \left| \frac{\partial v_{mi}}{\partial y_r} \right| \left| \frac{\partial N}{\partial y_l} \right| \left| \frac{v_{mi}}{N^2} \right| dy$$

$$\leq C \int_\Omega \left| \frac{\partial v_{mi}}{\partial y_r} \right| \left| \frac{\partial N}{\partial y_l} \right| |v_{mi}| \, dy \text{ using } \nabla N \in L^2(0, T; L^\infty(\Omega))$$

$$\leq C \|\nabla N\|_{L^\infty(\Omega)} \int_\Omega \left| \frac{\partial v_{mi}}{\partial y_r} \right| |v_{mi}| \, dy \leq C \|\nabla N\|_{L^\infty(\Omega)} \|v_m\| \, |v_m| \leq$$

$$\leq \varepsilon \|v_m\|^2 + C_\varepsilon \|\nabla N\|_{L^\infty(\Omega)}^2 \left| \frac{v_m}{\sqrt{N}} \right|^2$$

$$\left|\frac{1}{2}e\left(t;v_m,v_m\right)\right| \leq \int_\Omega |v_{mi}| \left|\frac{N'}{N^2}\right| |v_{mi}|\, dy \leq$$
$$\leq C\int_\Omega \left|N'\right| |v_m|^2\, dy \leq C\left\|N'\right\|_{L^\infty(\Omega)} \left|\frac{v_m}{\sqrt{N}}\right|^2$$
$$|d\left(t;v_m,v_m\right)| \leq \int_\Omega \frac{|G\left(N\right)|}{N} |v_m|\, |v_m|\, dy \leq C\left|\frac{v_m}{\sqrt{N}}\right|^2$$
$$\text{or } \ |d\left(t;v_m,v_m\right)| = \left|\sqrt{\frac{G\left(N\right)}{N}}v_m\right|^2.$$

Replacing the last six inequalities in (49), we get

$$\frac{1}{2}\frac{d}{dt}\left|\frac{v_m\left(t\right)}{\sqrt{N}}\right|^2 + \mu a_0\left\|v_m\right\|^2 + \mu\left|\sqrt{\frac{G\left(N\right)}{N}}v_m\right|^2 \leq C_\varepsilon\left\|g\right\|_{H^{-1}}^2 + \varepsilon\left\|v_m\right\|^2 +$$
$$+C\left\|N'\right\|_{L^\infty(\Omega)}\left|\frac{v_m}{\sqrt{N}}\right|^2 + \varepsilon\left\|v_m\right\|^2 + C\left\|\nabla N\right\|_{L^\infty(\Omega)}^2\left|\frac{v_m}{\sqrt{N}}\right|^2 + \varepsilon\left\|v_m\right\|^2 +$$
$$+C_\varepsilon\left|\frac{v_m}{\sqrt{N}}\right|^2$$

$$\frac{d}{dt}\left|\frac{v_m\left(t\right)}{\sqrt{N}}\right|^2 + 2\left(\mu a_0 - 3\varepsilon\right)\left\|v_m\right\|^2 + 2\mu\left|\sqrt{\frac{G\left(N\right)}{N}}v_m\right|^2 \leq C_\varepsilon\left\|g\right\|_{H^{-1}}^2 +$$
$$+C\left\|N'\right\|_{L^\infty(\Omega)}\left|\frac{v_m}{\sqrt{N}}\right|^2 + C\left\|\nabla N\right\|_{L^\infty(\Omega)}^2\left|\frac{v_m}{\sqrt{N}}\right|^2 + C_\varepsilon\left|\frac{v_m}{\sqrt{N}}\right|^2.$$

Take $\varepsilon > 0$ such that $\mu a_0 - 3\varepsilon > 0$, then

$$\frac{d}{dt}\left|\frac{v_m\left(t\right)}{\sqrt{N}}\right|^2 + \left\|v_m\right\|^2 \leq C\left\|g\right\|_{H^{-1}}^2 + C\left\|N'\right\|_{L^\infty(\Omega)}\left|\frac{v_m}{\sqrt{N}}\right|^2$$
$$C\left(\left\|\nabla N\right\|_{L^\infty(\Omega)}^2 + 1\right)\left|\frac{v_m}{\sqrt{N}}\right|^2$$

or

$$\frac{d}{dt}\left|\frac{v_m\left(t\right)}{\sqrt{N}}\right|^2 + \left\|v_m\right\|^2 \leq C\left\|g\right\|_{H^{-1}}^2 + \Phi\left(t\right)\left|\frac{v_m}{\sqrt{N}}\right|^2, \tag{50}$$

where $\Phi(t) = C\left\|N'\right\|_{L^\infty(\Omega)} + C\left(\left\|\nabla N\right\|_{L^\infty(\Omega)}^2 + 1\right)$. Using Gronwall and observing that Φ is integrable in $[0,T]$, we get

$$\left|\frac{v_m\left(t\right)}{\sqrt{N}}\right|^2 \leq \left(\exp\int_0^t \Phi\left(s\right)ds\right)\left(\left|\frac{v_m\left(0\right)}{\sqrt{N\left(0\right)}}\right|^2\right) + \int_0^t C\left\|g\right\|_{H^{-1}}^2 ds \leq C$$

Hence $v_m \in L^\infty(0,T;H)$. Integrating expression (50) in $[0,t]$ with $t<T$ we have $v_m(t) \in L^2(0,T;V)$, therefore

$$\left(\frac{v_m(t)}{\sqrt{N}}\right) \text{ is uniformly bounded in } L^\infty\left(0,T;L^2(\Omega)\right) \text{ and} \quad (51)$$
$$v_m(t) \text{ is uniformly bounded in} L^2(0,T;V)$$

$$\int_0^t \left|\nabla\left(\frac{v_m}{N}\right)\right|_{L^2}^2 dt = \int_0^t \left|\frac{\nabla v_m}{N} + v_m \nabla\left(\frac{1}{N}\right)\right|_{L^2}^2 dt \leq$$
$$C + C\int_0^t |\nabla N|^2 dt \text{ then } \frac{v_m}{N} \in L^2(0,T;H_0^1(\Omega)).$$

Second Estimates

Let $P_m : H \to V_m$ be the orthogonal projection of H onto V_m, that is,

$$P_m\varphi = \sum_{j=1}^m (\varphi, w_j)\, w_j.$$

Note that $P_m \in \pounds(V_s, V_s)$. In fact, since V_s is dense in H and $V_s \hookrightarrow V \hookrightarrow H$, we can restrict P_m to the space V_s for our estimates. Consider the orthonormal basis (w_j) and $\left(\frac{w_j}{\sqrt{\lambda_j}}\right)$ of H and V_s, respectively. Then, using (46), we get

$$\|P_m\|_{\pounds(V_s,V_s)} = \sup_{\|\varphi\|_{V_s}\leq 1} \|P_m\varphi\|_{V_s} = \sup_{\|\varphi\|_{V_s}\leq 1} \left\| \sum_{j=1}^m \left(\left(\varphi, \frac{w_j}{\sqrt{\lambda_j}}\right)\right)_{V_s} \frac{w_j}{\sqrt{\lambda_j}} \right\|_{V_s} =$$

$$= \sup_{\|\varphi\|_{V_s}\leq 1} \sqrt{\sum_{j=1}^m \left|\left(\left(\varphi, \frac{w_j}{\sqrt{\lambda_j}}\right)\right)_{V_s}\right|^2} \leq \sup_{\|\varphi\|_{V_s}\leq 1} \sqrt{\sum_{j=1}^\infty \left|\left(\left(\varphi, \frac{w_j}{\sqrt{\lambda_j}}\right)\right)_{V_s}\right|^2} = 1,$$

therefore

$$\|P_m\|_{\pounds(V_s,V_s)} \leq 1,$$

thus, by standard arguments:

$$\|P_m^*\|_{\pounds(V_s',V_s')} \leq 1.$$

Observe that

$$P_m\left(\frac{v_m}{N}\right)' = \sum_{j=1}^m \left(\frac{v_m'}{N} - v_m\frac{N'}{N^2}, w_j\right) w_j = \sum_{j=1}^m \left(\sum_{i=1}^m \frac{h_{im}'}{N}(w_i,w_j)w_j -$$
$$- \sum_{i=1}^m h_{im}\frac{N'}{N^2}(w_i,w_j)\Big)w_j = \sum_{j=1}^m \frac{h_{jm}'}{N}w_j - \sum_{j=1}^m h_{im}\frac{N'}{N^2}w_j = \frac{v_m'}{N} - v_m\frac{N'}{N} = \left(\frac{v_m}{N}\right)' \quad (52)$$

Hence, if we multiply the equation (47) by w_j and sum for $j = 1, \ldots, m$, we have

$$\sum_{j=1}^{m} \left(\frac{v_m'(t)}{N}, w_j \right) w_j + \mu \sum_{j=1}^{m} a(t; v_m(t), w_j) w_j + \sum_{j=1}^{m} b(t; v_m(t), v_m(t), w_j) w_j + \\ + \sum_{j=1}^{m} c(t; v_m(t), w_j) w_j + \mu \sum_{j=1}^{m} d(t; v_m(t), w_j) w_j = \sum_{j=1}^{m} (g, w_j) w_j$$

Since

$$\sum_{j=1}^{m} \left(\frac{v_m'(t)}{N}, w_j \right) w_j = \sum_{j=1}^{m} \left(\left(\frac{v_m}{N} \right)' + v_m \frac{N'}{N^2}, w_j \right) w_j = P_m \left(\frac{v_m}{N} \right)' \\ + \sum_{j=1}^{m} e(t; v_{mj}, w_j) w_j$$

and using the notation of lemmas 2 and 3

$$P_m \left(\frac{v_m}{N} \right)' + < E(t) v_m, w_j > w_j + \sum_{j=1}^{m} (\mu < A(t) v_m, w_j > w_j + < B(t) v_m, w_j > w_j) \\ + \sum_{j=1}^{m} (< C(t) v_m, w_j > w_j + \mu < D(t) v_m, w_j > w_j) = \sum_{j=1}^{m} (g, w_j) w_j;$$

hence, and from (52),

$$\left(\frac{v_m}{N} \right)' = \sum_{j=1}^{m} < g - (\mu A(t) + B(t) + C(t) + \mu D(t) + E(t)) v_m, w_j > w_j,$$

since $g - \mu A(t) v_m - B(t) v_m - C(t) v_m - \mu D(t) v_m - E(t) v_m \in V_s'$, we have

$$\left(\frac{v_m}{N} \right)' = P_m^* (g - \mu A(t) v_m - B(t) v_m - C(t) v_m - \mu D(t) v_m - E(t) v_m). \tag{53}$$

Then taking the norm of $\left(\frac{v_m}{N} \right)'$ in V_s', applying the triangular inequality, using $\|P_m^*\|_{\mathcal{L}(V_s', V_s')} \leq 1$ and applying the Young's inequality repeatedly, it follows that

$$\left\| \left(\frac{v_m}{N} \right)' \right\|_{V_s'(\Omega)}^2 \leq 4 \left(\|\mu A(t) v_m\|_{V_s'(\Omega)}^2 + \|B(t) v_m\|_{V_s'(\Omega)}^2 + \|C(t) v_m\|_{V_s'(\Omega)}^2 \right) + \\ + 4 \left(\|\mu D(t) v_m\|_{V_s'(\Omega)}^2 + \|E(t) v_m\|_{V_s'(\Omega)}^2 + \|g\|_{V_s'(\Omega)}^2 \right).$$

Now we estimate each term of the right hand side of the last expression:

- from part (iii) of lemma 2,

$$\|\mu A(t) v_m\|_{V_s'} = \sup_{\|w\|_{V_s} \leq 1} |< \mu A(t) v_m, w >| \leq C \|v_m\|_{V_s'} \|w\|_{V_s} \leq C \|v_m\|_{V_s'},$$

and

$$\int_0^T \|\mu A(t)\, v_m\|^2_{V_s'(\Omega)}\, ds \le C \int_0^T \|v_m\|^2_{V_s'(\Omega)}\, ds \le C \int_0^T \|v_m\|^2_{V(\Omega)}\, ds < \infty,$$

since, from (51), $v_m \in L^2(0,T;V)$ and $V \hookrightarrow V_s'$.

- from part (iii) of lemma 4

$$\|B(t)\, v_m\|_{V_s'(\Omega)} \le C \|v_m\|^2_{(L^p(\Omega))^n}, \quad \text{where } \frac{1}{p} = \frac{1}{2} - \frac{1}{2n}.$$

Then

$$\int_0^T \|B(t)\, v_m\|^2_{V_s'(\Omega)}\, ds \le \int_0^T C \|v_m\|^4_{(L^p(\Omega))^n}\, ds.$$

By (51) , it holds that $v_m \in L^2(0,T;V) \cap L^\infty(0,T;H)$. Thus, from part (viii) of lemma 4, $v_m \in L^4(0,T;(L^p(\Omega))^n)$. Hence and from the last inequality,

$$\int_0^T \|B(t)\, v_m\|^2_{V_s'(\Omega)}\, ds < \infty$$

- from part (iv) of lemma 4

$$\|C(t)\, v_m\|_{V_s'(\Omega)} \le |C(t)\, v_m| \le C \|v_m\|_{V_s'(\Omega)},$$

that implies

$$\int_0^T \|C(t)\, v_m\|^2_{V_s'(\Omega)}\, ds \le \int_0^T C \|v_m\|^2_{V_s'(\Omega)}\, ds < \infty.$$

- $\|g\|_{V_s'(\Omega)} \le \|g\|_{H(\Omega)}$.

Therefore

$$\left(\frac{v_m}{N}\right)' \text{ is bounded in } L^2(0,T;V_s'(\Omega)). \tag{54}$$

Define

$$W = \left\{ v_m; \frac{v_m}{N} \in L^2\left(0,T;H_0^1(\Omega)\right) \text{ and } \left(\frac{v_m}{N}\right)' \in L^2(0,T;V_s'(\Omega)) \right\},$$

with the norm $\|v_m\|_{L^2(0,T;V(\Omega))} + \|v_m'\|_{L^2(0,T;V_s'(\Omega))}$. Since $V \hookrightarrow H$ and $H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow V_s'(\Omega)$ compactly, by Aubim-Lions theorem $W \hookrightarrow L^2(0,T;L^2(\Omega))$ compactly. From this and by (51) there exists a subsequence of $\left(\frac{v_m}{N}\right)$, still denoted by $\left(\frac{v_m}{N}\right)$, and a function v such that :

$$v_m \rightharpoonup v \text{ weakly in } L^2(0,T;V) \tag{55}$$

$$\frac{v_m}{N} \rightharpoonup \frac{v}{N} \text{ weakly in } L^2\left(0,T;H_0^1(\Omega)\right) \tag{56}$$

$$v_m \rightharpoonup^* v \text{ weakly-star in } \quad L^\infty(0,T;H) \tag{57}$$

$$\frac{v_m}{N} \rightharpoonup^* \frac{v}{N} \text{ weakly-star in } \quad L^\infty\left(0,T;L^2(\Omega)\right) \tag{58}$$

$$\begin{cases} \frac{v_m}{N} \to \frac{v}{N} \text{ strongly in } L^2(0,T;L^2(\Omega)) \\ \left(\frac{v_{n_i}}{N} \to \frac{v}{N} \text{ strongly in } L^2(0,T;L^2(\Omega))\right) \text{ and } a.e. \text{ in } Q \end{cases} \tag{59}$$

$$\left(\frac{v_m}{N}\right)' \rightharpoonup \left(\frac{v}{N}\right)' \text{ weakly in } \quad L^2(0,T;V_s'(\Omega)) \tag{60}$$

Let $\Psi(t)$ be in $C_0^\infty(0,T)$; multiplying the equation (47) by $\Psi(t)$ and integrating with respect to t, we have

$$\int_0^T \left(\left(\frac{v_m}{N}\right)', \Psi w_j \right) dt + \int_0^T \mu a(t; v_m, \Psi w_j)\, dt + \int_0^T b(t; v_m, v_m, \Psi w_j)\, dt +$$
$$\int_0^T c(t; v_m, w_j)\, \Psi dt + \int_0^T \mu d(t; v_m, \Psi w_j)\, dt + \int_0^T \mu e(t; v_m, \Psi w_j)\, dt = \int_0^T (g, \Psi w_j)\, dt \ .$$

Taking the limit for each term, by (58)

$$\int_0^T \left(\left(\frac{v_m}{N}\right)', \Psi w_j \right) dt = -\int_0^T \left(\frac{v_m}{N}, \Psi' w_j \right) dt \to \int_0^T \left(\frac{v}{N}, \Psi' w_j \right) dt \ ,$$

$$\int_0^T a(t; v_m, \Psi w_j)\, dt = \int_0^T \int_\Omega \beta_{lj}\beta_{rj} \left(\frac{\partial v_{mi}}{\partial y_r}, \Psi \frac{\partial}{\partial y_r}\left(\frac{w_{ji}}{N}\right) \right) dy dt \to \int_0^T a(t; v, \Psi w_j)\, dt$$

by (56), due to lemma 3.2 TEMAN [10], pg 285.

$$\int_0^T c(t; v_m, w_j)\, \Psi dt = \int_0^T \int_\Omega \beta'_{lr}(t)\, \alpha_{rj}(t)\, y_j \frac{\partial v_{mi}(y)}{\partial y_l} \frac{w_{ji}(y)}{N} \Psi dy dt$$

$$F(t) = \beta'_{lr}(t)\, \alpha_{rj}(t)\, y_j \frac{w_{ji}(y)}{N} \in H_0^1(\Omega) \text{ and } G_m(t) = \frac{\partial v_{mi}(y)}{\partial y_l} \in L^2(\Omega) \text{ so}$$

$$\int_0^T c(t; v_m, w_j)\, \Psi dt = \int_0^T (F(t),\ G_m(t))\, dt = \int_0^T \langle F(t),\ G_m(t) \rangle\, dt$$

converges (see pg 248 TEMAN [10]).

$$\int_0^T d(t; v_m, \Psi w_j)\, dt \to \int_0^T \int_\Omega G(N)\, v_i(y)\, \frac{w_{ji}(y)}{N} \Psi dy dt$$

due to (59)

$$\int_0^T \mu e\left(t; v_m, \Psi w_j\right) dt \to \int_0^T \int_\Omega v_i\left(y\right) \frac{N^{'}}{N^2} w_{ji}\left(y\right) \Psi dy dt$$

$$\text{since } \left| \int_0^T \int_\Omega \left(v_{mi}\left(y\right) - v_i\left(y\right)\right) \frac{N^{'}}{N^2} w_{ji}\left(y\right) \Psi dy dt \right| \to 0$$

due to (57) with the following statement: $\frac{N^{'}}{N^2} w_{ji}\left(y\right) \Psi \in L^1\left(0, T, L^2\left(\Omega\right)\right)$ since

$$\int_0^T \left\| \frac{N^{'}}{N^2} w_{ji}\left(y\right) \Psi \right\| dt \leq C \int_0^T \left\| N^{'} \right\|_\infty \left\| w_{ji}\left(y\right) \right\| dt < \infty \text{ from } N^{'} \in L^1\left(0, T, L^\infty\left(\Omega\right)\right).$$

Therefore, the theorem holds. The initial condition $v\left(0\right) = v_0$ is achieved from (55) and (60), that is, $v_m \rightharpoonup v$ weakly in $L^2\left(0, T, V_s^{'}\right)$ and $v_m^{'} \rightharpoonup v$ weakly-* in $L^2\left(0, T, V_s^{'}\right)$ hence $v \in W^{1,2}\left(0, T, V_s^{'}\right) \Rightarrow u \in C\left(0, T, V_s^{'}\right)$ and $V_s \subset H \subset V_s^{\prime}$, $v \in V$ and $w \in H, \left\langle v, w \right\rangle_{V_s V_S^{'}} = \left(v, w\right)_H$.

□

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