ON EMBEDDING MODELS OF ARITHMETIC INTO REDUCED POWERS

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1 Introduction

In 1934, Thoralf Skolem[1] constructed a family of nonstandard models of true arithmetic without making use of the completeness or compactness theorems for first order logic. Skolem's models are substructures of a certain structure $\mathcal{N}$ that will be the central topic of this paper. His discovery, from our point of view, was that $\mathcal{N}$ contains nonstandard models of true arithmetic. This is interesting because $\mathcal{N}$ has a much simpler definition than any nonstandard model of true arithmetic constructed by means of the completeness theorem. The domain of $\mathcal{N}$ and its operations occur naturally in mathematics.

We now define $\mathcal{N}$ and for historical reasons describe Skolem's construction. Let $\mathbb{N}^{\mathbb{N}}$ be all functions from the natural numbers to the natural numbers. We define an equivalence relation on $\mathbb{N}^{\mathbb{N}}$ as follows: Two functions f and g are equivalent if the equation $f(x) = g(x)$ holds for all but finitely many natural numbers x. Let $[f]$ be the equivalence class of the function f. We add and multiply equivalence classes by the rules

$$[f] + [g] = [f + g] \text{ and } [f] \cdot [g] = [f \cdot g].$$

Equivalence classes are partially ordered by the relation

$$[f] \leq [g] \text{ iff for all but finitely many } x,\ f(x) \leq g(x).$$

The structure $\mathcal{N}$ has domain all equivalence classes $[f]$, and is equipped with the addition, multiplication and order relation we have just defined.

Skolem found nonstandard models of true arithmetic inside $\mathcal{N}$ by constructing a function g from the natural numbers to the natural numbers with the

[1] See [12].

following property: If S is any set of natural numbers defined by an arithmetic formula, then either $g(x)$ is an element of S for all but finitely many x, or $g(x)$ fails to be an element of S for all but finitely many x. Assume that we have such a function g. Let $f_1, f_2, \dots$ be all arithmetic functions from the natural numbers to the natural numbers. Skolem showed that the set of all equivalence classes of the form $[f_i \circ g]$ gives a substructure M of $\mathcal{N}$ satisfying true arithmetic. For a very similar argument, see the statement and proof of Theorem 32.

Skolem's models do not exhaust the isomorphism types of countable models of true arithmetic. Thus it is reasonable to ask if there are other countable models of true arithmetic inside $\mathcal{N}$, not isomorphic to any of those given by Skolem's construction. Skolem did not pursue this problem, perhaps because he did not have the structure $\mathcal{N}$ specifically in view. In the early 1970's, Stanley Tennenbaum proved that every countable model of true arithmetic is present, up to isomorphism, in $\mathcal{N}$, via mappings which embed the standard part of the model canonically, i.e. via mappings which send each standard integer k to the equivalence class of the constant function $< k, k, \dots >$. He also proved a second, analogous theorem for nonnegative parts of discretely ordered rings. (See Theorem 4.)

A model of arithmetic contained in $\mathcal{N}$ is a more concrete object than one given by the completeness theorem. The elements of such a model are, on the one hand, integer-valued functions, and on the other hand, objects that belong to a certain element type in a model of arithmetic. What is the connection between the two? For example, if a function f takes on only prime values, and if $[f]$ belongs to a model of true arithmetic contained in $\mathcal{N}$, must $[f]$ be prime in that model? The study of the connection between satisfaction and *componentwise satisfaction*, i.e. satisfaction in $\mathcal{N}$, is one of the two main topics of this paper. The other is this: What numerical properties must a function have in order to belong to a model of a given subtheory of true arithmetic? For example, can the identity function belong to a model of true arithmetic? As we shall see, the answer is no. We will see that the functions that belong to particular subtheories of true arithmetic are of a kind very similar to the function g of Skolem's proof.[2]

[2] See the definition of cohesiveness in Section 6.

2 Preliminaries

Let LA, the language of arithmetic, be the first order language with non-logical symbols $+, \cdot, 0, 1, \leq$. $\mathbb{N}$ denotes the standard LA structure.

We shall be concerned with the following theories: The $\forall_1$-$Th(\mathbb{N})$, Π_1-$Th(\mathbb{N})$, and the Π_2-$Th(\mathbb{N})$, defined respectively as the set of all $\forall_1$, Π_1 and Π_2 formulas true in the standard LA structure $\mathbb{N}$. $Th(\mathbb{N})$ is the theory *true arithmetic*, i.e., the set of all the first order sentences in the language of arithmetic true in the standard structure $\mathbb{N}$.

We shall also be concerned with the theory PA^-, which is the theory of nonnegative parts of discretely ordered rings. For the axioms of PA^- as well as interesting examples of these models see [7].

Also, we mean by the MRDP Theorem the result of Matiyasevich, Robinson, Davis and Putnam, which states that every recursively enumerable subset of the natural numbers has a Diophantine definition.

Finally, we assume throughout that any embedding of a model of arithmetic we mention has the following property: the standard structure $\mathbb{N}$ is embedded into $\mathcal{N}$ canonically, i.e. under the embedding the standard integers k are identified with the equivalence classes of the constant functions $[< k, k, \ldots >]$.

3 The Basic Construction

Let LA be as above. If $\mathbb{N}$ is the standard LA structure, let $\mathcal{N}$ be the reduced power (of LA structures) $\mathbb{N}^\omega/\mathcal{F}$, where $\mathcal{F}$ is the cofinite filter in the boolean algebra of subsets of $\mathbb{N}$. Let $\mathbb{A}$ be the standard LA structure with domain all nonnegative real algebraic numbers, and let $\mathcal{A}$ be the reduced power $\mathbb{A}^\omega/\mathcal{F}$. We shall see that every countable model of PA^- is contained, up to isomorphism, in $\mathcal{A}$. Moreover, a countable model of PA^- appears, up to isomorphism, as a substructure of $\mathcal{N}$ if and only if it is Diophantine correct, i.e., a model of the $\forall_1$-$Th(\mathbb{N})$.

If f is a function from $\mathbb{N}$ to $\mathbb{N}$, let $[f]$ denote the equivalence class of f in $\mathcal{N}$. We use a similar notation for $\mathcal{A}$. When no confusion is possible, we will use f and $[f]$ interchangeably. Central to this paper is the notion of satisfaction in $\mathcal{N}$, which we refer to via the following special

Definition 1 *If $\phi(x_1, \ldots, x_n)$ is an LA formula, and $f_1, \ldots, f_n$ are in $\mathcal{N}$, then* *we say* $\phi(f_1, \ldots f_n)$ *holds componentwise or is true componentwise if, for all sufficiently large* i, $\mathbb{N} \models \phi(f_1(i), \ldots, f_n(i))$.

As we will see, there are substructures of $\mathcal{N}$ for which componentwise truth and the usual satisfaction relation are identical. Let M be such a substructure. Then in M, the LA formulas satisfied by an element $[f]$ are completely determined by the LA formulas satisfied by the numbers $f(1), f(2), \ldots$. For example, $[f]$ is prime in M iff for large enough n, the numbers $f(n)$ are standard primes. Note that M must satisfy $Th(\mathbb{N})$, since all sentences holding in $\mathbb{N}$ hold componentwise. Now in any model M of $Th(\mathbb{N})$, a total recursive function g from $\mathbb{N}$ to $\mathbb{N}$ has a unique continuation g^M to M, via any Σ_1 formula defining the graph of g over $\mathbb{N}$. Because satisfaction and componentwise truth coincide in M, we get a very concrete picture of the behavior of g^M: If $[f]$ is in M, then $g^M([f]) = [g \circ f]$.

Such a strong relation between truth and componentwise truth is far from typical. We shall be concerned with the properties of substructures of $\mathcal{N}$ which imply and are implied by such relations. We shall characterize the LA theories that prove the MRDP Theorem in terms of the componentwise behavior of Δ_0 formulas in models of those theories. We shall establish a connection between the componentwise behavior of LA formulas, and the preservation of those formulas in extensions. And we shall also give characterizations of models of PA^-, Π_1-$Th(\mathbb{N})$, and Π_2-$Th(\mathbb{N})$ in terms of componentwise behavior.

We begin by presenting the two embedding theorems of Stanley Tennenbaum, which represent countable models of PA^- by means of sequences of real numbers. The proofs are slightly modified versions of the proofs communicated to us by him.

Theorem 2 (Tennenbaum) *Let* M *be a countable Diophantine correct model of* PA^-. *Then* M *can be embedded in* $\mathcal{N}$.

Proof. Let $m_1, m_2, \ldots$ be the distinct elements of M. Let $P_1, P_2, \ldots$ be all polynomial equations over $\mathbb{N}$ in the variables $x_1, x_2, \ldots$ such that $M \models P_i(x_1/m_1, x_2/m_2, \ldots)$. Each system of equations $P_1 \bigwedge \cdots \bigwedge P_n$ has a solution in M. Thus, by Diophantine correctness, there is a sequence of natural numbers $v_1(n), v_2(n), \ldots$ for which

$$\mathbb{N} \models P_1 \bigwedge \cdots \bigwedge P_n(x_1/v_1(n), x_2/v_2(n), \ldots).$$

Note that if the variable x_i doesn't appear in $P_1 \bigwedge \cdots \bigwedge P_n$, then the choice of $v_i(n)$ is completely arbitrary.

Our embedding $h : M \longrightarrow \mathcal{N}$ is given by:

$$m_i \longmapsto [\lambda n. v_i(n)].$$

In the figure below, the i-th row is the solution in integers to $P_1 \bigwedge \cdots \bigwedge P_n$, and the i-th column "is" $h(m_i)$.

	m_1	m_2	$\dots$	m_n	$\dots$
P_1	$v_1(1)$	$v_2(1)$	$\dots$	$v_n(1)$	$\dots$
P_2	$v_1(2)$	$v_2(2)$	$\dots$	$v_n(2)$	$\dots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	
P_n	$v_1(n)$	$v_2(n)$	$\dots$	$v_n(n)$	$\dots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	

Note that if m_e is the element 0^M of M, then the polynomial equation $x_e = 0$ appears as one of the P's. It follows that, for n sufficiently large, $v_e(n) = 0$. Thus $h(0^M)$ is the equivalence class of the zero function. Similarly, h maps every standard integer of M to the class of the corresponding constant function.

We show that h is a homomorphism. Suppose $M \models m_i + m_j = m_k$. Then the polynomial $x_i + x_j = x_k$ must be one of the P's, say P_r. If $n \geq r$, then by construction $v_i(n) + v_j(n) = v_k(n)$. Hence $h(m_i) + h(m_j) = h(m_k)$, as required. A similar argument works for multiplication. Suppose $M \models m_i \leq m_j$. By an axiom of PA^-, for some k, $M \models m_i + m_k = m_j$. Thus, as we've shown, $h(m_i) + h(m_k) = h(m_j)$. It follows from the definition of the relation $\leq$ in $\mathcal{N}$ that $h(m_i) \leq h(m_j)$.

To see that h is one to one, suppose that $m_i \neq m_j$. Since in models of PA^- the order relation is total, we may assume that $m_i < m_j$. Again by the axioms of PA^-, we can choose m_k such that $m_i + m_k + 1 = m_j$. As we have shown, $h(m_i) + h(m_k) + h(1) = h(m_j)$. Since h(1) is the class of the constant function 1, it follows that $h(m_i) \neq h(m_j)$.

□

Corollary 3 *Let M be a countable model of the $\forall_1$-$Th(N)$. Then M can be embedded in $\mathcal{N}$.*

Proof. The models of the $\forall_1$-$Th(N)$ are precisely the substructures of models of $Th(N)$. Thus, M extends to a model of PA^-, which can be embedded in $\mathcal{N}$

as in Theorem 2.

□

Before turning to the theorem for the non-Diophantine correct case, we observe first that the given embedding depends upon a particular choice of enumeration $m_1, m_2, \ldots$ of M, since different enumerations will in general produce different polynomials. We also note that different choices of solution yield different embeddings. Also, as we shall see below, we need not restrict ourselves to Diophantine formulas: we can carry out the construction for LA formulas of any complexity which hold in M.

We state next the non-Diophantine correct case of the theorem:

Theorem 4 (Tennenbaum) *Let M be a countable model of PA^-. Then M can be embedded in $\mathcal{A}$.*

Proof. Given an enumeration $m_1, m_2, \ldots$ of M, we form conjunctions of polynomial equations P_n exactly as before. We wish to produce solutions of $P_1 \bigwedge \cdots \bigwedge P_n$ in the nonnegative algebraic reals for each n. We proceed as follows: The model M can be embedded in a real closed field F by a standard construction. (Embed M in an ordered integral domain, then form the (ordered) quotient field, and then the real closure. See [9].) Choose k so large that $x_1, \ldots, x_k$ are all the variables that occur in the conjunction $P_1 \bigwedge \cdots \bigwedge P_n$. The sentence $\exists x_1 \ldots x_k (P_1 \bigwedge \cdots \bigwedge P_n \bigwedge x_1 \geq 0 \bigwedge x_2 \geq 0 \cdots \bigwedge x_k \geq 0)$ is true in M, hence in F. It is a theorem of Tarski that the theory of real closed fields is complete. (See [4]). Thus, this same sentence must be true in the field of real algebraic numbers. This means we can choose nonnegative algebraic real numbers $v_1(n), v_2(n) \ldots$ satisfying the conjunction $P_1 \bigwedge \cdots \bigwedge P_n$. Let $h : M \longrightarrow \mathcal{A}$ be given by

$$m_i \longmapsto [\lambda n. v_i(n)].$$

The proof that h is a homomorphism, and furthermore an embedding, proceeds exactly as before, once we note that the equivalence classes all consist of nonnegative sequences of real algebraic numbers.

□

Remark 5 *Under any of the embeddings given above, if $M \models PA^-$ then nonstandard elements of M are mapped to equivalence classes of functions tending to infinity. Why? If f is a function in the image of M, and f does not tend* *to infinity, then choose an integer k such that f is less than k infinitely often. Since $M \models PA^-$, either $[f] \leq [k]$ or $[k] \leq [f]$. The second alternative contradicts the definition of $\leq$ in $\mathcal{N}$. Hence $[f] \leq [k]$, i.e., $[f]$ is standard.*

Remark 6 *Let F be a countable ordered field. Then F is embedded in $\mathbb{R}^\omega/\mathcal{F}$, where $\mathbb{R}$ is the field of real algebraic numbers. The proof is mutatis mutandis the same as in Theorem 4, except that due to the presence of negative elements we must demonstrate differently that the mapping obtained is one to one. But this must be the case, since every homomorphism of fields has this property. (See [9].)*

Remark 7 *For any pair of LA structures A and B satisfying PA^-, if A is countable and if A satisfies the $\forall_1$-$Th(B)$ then there is an embedding of A into $B^\omega/\mathcal{F}$. In particular, if M is a model of PA^-, then every countable extension of M satisfying the $\forall_1$-$Th(M)$ can be embedded in $M^\omega/\mathcal{F}$.*

4 Componentwise Behavior and Open Formulas

Let M be a substructure of $\mathcal{N}$, and let ϕ be a formula with parameters from M. If ϕ holds in M we cannot conclude that ϕ holds componentwise. For example, let $f : N \longrightarrow N$ be the function taking even numbers to 0 and odd numbers to 1. Let M be the substructure of $\mathcal{N}$ generated by $[f]$. If ϕ is the formula $[f] \neq [0]$ then ϕ holds in M, but does not hold componentwise. It is just as easy to find examples where the converse implication fails. What, then, is the relation between componentwise truth and truth in a substructure of $\mathcal{N}$? We shall study classes of formulas ϕ and classes of substructures M of $\mathcal{N}$ for which componentwise truth and satisfaction are connected in interesting ways.

Let M be a substructure of $\mathcal{N}$. An atomic formula with parameters from M holds in M iff it holds in $\mathcal{N}$ iff it holds componentwise. From this, it follows that an atomic formula prefaced by a string of existential quantifiers that holds in M will hold componentwise.

As the example above shows, it is not in general true that open formulas that are true in M hold componentwise. However, we have

Proposition 8 *Suppose M is a substructure of $\mathcal{N}$ that satisfies PA^-. Then an open formula with parameters from M is true in M iff it holds componentwise.*

Proof. Suppose ϕ is an open formula with parameters from M, and suppose ϕ holds in M. The formula ϕ is equivalent, in any model of PA^-, to an atomic formula prefaced by a string of existential quantifiers. (Using the axioms for a total order, and the axiom $\forall y \forall x \leq y \exists z x + z = y$, one converts inequalities into equations prefaced by strings of existential quantifiers. One then combines conjunctions and disjunctions of equations into a single equation.) Thus, by a previous remark, because ϕ holds in M it

must hold componentwise. Conversely, if ϕ holds componentwise, then the open formula $\sim \phi$ cannot hold in M, else, as we have already established, it would hold componentwise. Hence ϕ must hold in M.

□

From this, we have

Proposition 9 *Suppose M is a substructure of $\mathcal{N}$ satisfying PA^-, and ϕ is a formula with parameters from M. If ϕ is $\exists_1$ and $M \models \phi$ then ϕ holds componentwise. If ϕ is $\forall_1$ and ϕ holds componentwise then $M \models \phi$.*

Proof. The first assertion follows from Proposition 8. As for the second, suppose ϕ is $\forall_1$ and ϕ fails to hold in M. Then the $\exists_1$ formula $\sim\phi$ holds in M, hence holds componentwise. Hence ϕ cannot hold componentwise.

□

We note that not every substructure of $\mathcal{N}$ satisfies PA^-, or the $\forall_1$-$Th(\mathbb{N})$ for that matter. For example, $\mathcal{N}$ itself fails to satisfy the $\forall_1$ sentence asserting that the order relation is total. Moreover, there are substructures of $\mathcal{N}$ satisfying the $\forall_1$-$Th(\mathbb{N})$ but not satisfying PA^-. An example is the substructure M of $\mathcal{N}$ generated by the identity function f on $\mathbb{N}$. To see that $M \models \forall_1$-$Th(\mathbb{N})$, we observe that M is isomorphic to $\mathbb{N}[x]$, the semiring of polynomials over $\mathbb{N}$ in the variable x. But this structure $\mathbb{N}[x]$ appears up to an isomorphism in any nonstandard model of $Th(\mathbb{N})$, as the substructure generated by a nonstandard element. This means M extends to a model of $Th(\mathbb{N})$. Since universal formulas are downward preserved, we have shown that $M \models \forall_1$-$Th(\mathbb{N})$. On the other hand, M fails to satisfy PA^- since, in M, f has no predecessor.

However, we have

Proposition 10 *If M is a substructure of $\mathcal{N}$ and $M \models PA^-$, then $M \models \forall_1$-$Th(\mathbb{N})$.*

Proof. Suppose $\phi(\vec{x})$ is open, and $N \models \forall\vec{x}\phi(\vec{x})$. Let $\vec{f}$ be a tuple from M. Then $\phi(\vec{f})$ holds componentwise. Thus, by Proposition 8, $\phi(\vec{f})$ holds in M. Since $\vec{f}$ was chosen arbitrarily from M, it follows that $M \models \forall\vec{x}\phi(\vec{x})$, as required.

□

For the next result, we need the following

Definition 11 *Let M be a substructure of $\mathcal{N}$. Let Φ be a set of LA formulas. We say that Φ behaves componentwise in M if for all tuples $\vec{f}$ from M, and for all formulas ϕ from Φ, $M \models \phi(\vec{f})$ iff $\phi(\vec{f})$ holds componentwise.*

For which substructures of $\mathcal{N}$ do the open formulas behave componentwise?

Theorem 12 (Kennedy-Raffer) [3] *Let M be a substructure of $\mathcal{N}$. Then the following are equivalent: (i) The open formulas behave componentwise in M. (ii) $M \models \forall_1$-$Th(\mathbb{N})$. (iii) M extends to a model of PA^- included in $\mathcal{N}$.*

Proof.

(i) implies (ii): Assume (i), and let ϕ be an open formula such that $\mathbb{N} \models \forall\vec{x}\phi$. Suppose it was the case that $M \models \exists\vec{x} \sim\phi$. Choose a tuple $\vec{f}$ from M such that $M \models\sim \phi(\vec{x}/\vec{f})$. Then the latter holds componentwise, so there is a tuple of integers $\vec{n}$ such that $\mathbb{N} \models\sim\phi(\vec{x}/\vec{n})$. But this contradicts $\mathbb{N} \models \forall\vec{x}\phi$.

(ii) implies (iii): Assume (ii). Let M' be

$$\{a \in \mathcal{N} : \exists b, c \in M \ s.t. \ a + b = c\}.$$

Then M' is a substructure of $\mathcal{N}$ and M' contains M. We will show that $M' \models PA^-$. As a substructure of $\mathcal{N}$, M' is automatically a partially ordered semiring[4]. So we have to check that the order on M' is total and discrete (meaning that two consecutive elements cannot bound a third), and that M' is closed under nonnegative differences. Note that $\forall_1$ sentences express that the order relation is total and discrete in M. First we will show that the ordering is total in M'. If a and b are in M', choose elements r, s, r' and s' in M such that $a + r = s$ and $b + r' = s'$. Then $a + r + s' = b + r' + s$. Now $r + s'$ and $r' + s$ are in M, so they are comparable: We will suppose that $r' + s \leq r + s'$. It follows

[3]Theorems attributed to Kennedy-Raffer are proved jointly with Sidney Raffer and are published here with his permission.

[4]A semiring is a structure of type $(+, \cdot, 0, 1)$ obeying all the axioms for a commutative ring except the one requiring the existence of additive inverses.

that the formula $a \leq b$ holds componentwise, hence holds in M'. This proves that M' is totally ordered. To prove discreteness, suppose (keeping the same notation) that $a \leq b \leq a+1$. From the relations $a+r=s$ and $b+r'=s'$ we conclude that $s+r' \leq s'+r \leq s+r'+1$. Since M is discretely ordered, it follows that either $s'+r=s+r'$, in which case $a=b$, or $s'+r=s'+r+1$, in which case $a+1=b$. This proves that the order on M' is discrete. As for nonnegative differences, suppose, (with the same notation) that $a \leq b$. Choose c in $\mathcal{N}$ so that $a+c=b$. We have to show that c is in M': But the equation $a+r+s'=b+r'+s$ implies that that $r+s'=c+r'+s$. Thus c is indeed in M'.

(iii) implies (i): Suppose M extends to a model M' of PA^- included in $\mathcal{N}$. Let ϕ be an open formula with parameters from M, and suppose ϕ holds in M. We have to check that ϕ holds componentwise. But this follows from Proposition 8, and the fact that ϕ holds in M'. Conversely, suppose ϕ holds componentwise. Then $\sim \phi$ cannot hold in M, else it would hold in M' and therefore would hold componentwise. Hence ϕ holds in M.

□

5 Componentwise Behavior and Σ_1 Formulas

The requirement that a substructure of $\mathcal{N}$ satisfy PA^- is not strong enough to insure that the Δ_0 formulas behave componentwise. For example, let $\mathbb{Z}[t]^+$ be the semiring of all polynomials in t over $\mathbb{Z}$ with positive leading coefficients. We order this semiring by the rule $p(t) \leq q(t)$ if the leading coefficient of $q(t)-p(t)$ is nonnegative. We then obtain an LA structure satisfying PA^-. If f is any function from $\mathbb{N}$ to $\mathbb{N}$, and if f tends to infinity, then $\mathbb{Z}[t]^+$ can be embedded in $\mathcal{N}$ via the map

$$p(t) \mapsto [|p(f)|],$$

where '$|\ |$' denotes absolute value. (The function $p(f)$ is eventually nonnegative. The point of the absolute value is to make sure it is always nonnegative.) For example, take f to be the identity function. Let $\phi(x)$ be the Δ_0 formula "x is not even and x is not odd." Then $\phi(f)$ holds in $\mathbb{Z}[t]^+$, but does not hold componentwise. On the other hand, we have:

Proposition 13 *If M is a substructure of $\mathcal{N}$, and $M \models MRDP + PA^-$, then the Δ_0 formulas behave componentwise in M.*

Before proving the proposition, we need the following

Definition 14 *We say that an LA structure M satisfies the MRDP Theorem if every Δ_0 formula is equivalent, over M, to an $\exists_1$ formula.*

Proof. Suppose, then, that $M \models MRDP$. Let $\phi(\vec{x})$ be a Δ_0 formula and let $\vec{f}$ be a tuple from M. Suppose $M \models \phi(\vec{f})$. Let $\psi(\vec{x})$ be the $\exists_1$ equivalent of $\phi(\vec{x})$ over M. Then $M \models \psi(\vec{f})$ and $\psi(\vec{f})$ holds componentwise. Now the sentence $\sim\forall\vec{x}(\psi(\vec{x}) \rightarrow \phi(\vec{x}))$ is logically equivalent to a Σ_1 formula. If it held in $\mathbb{N}$ then, by the well known "Σ_1-completeness" of PA^- it would hold in M, which it does not. (Every Σ_1 sentence true in $\mathbb{N}$ is provable in PA^-. See [7], chapter 3) Thus $\mathbb{N} \models \forall\vec{x}(\psi(\vec{x}) \rightarrow \phi(\vec{x}))$. It follows that $\phi(\vec{f})$ holds componentwise. In order to complete the proof, we now have to assume that $\phi(\vec{f})$ holds componentwise, and show that it holds in M. But, if it did not, then the above argument carried out with $\sim\phi(\vec{f})$ in place of $\phi(\vec{f})$ would give a contradiction.

□

If the Δ_0 formulas behave componentwise in $M \subseteq \mathcal{N}$, what can we say about $Th(M)$?

Proposition 15 *Suppose M is a substructure of $\mathcal{N}$ in which the Δ_0 formulas behave componentwise. Then $M \models \Pi_1$-$Th(\mathbb{N})$.*

Proof. Suppose $\phi(\vec{x})$ is a Δ_0 formula, and $\mathbb{N} \models \forall\vec{x}\phi(\vec{x})$. Choose $\vec{f}$ from M. Then $\phi(\vec{f})$ holds componentwise, hence it holds in M. But $\vec{f}$ in M was arbitrary. So $M \models \forall\vec{x}\phi(\vec{x})$

□

Next, we consider the componentwise behavior of $\exists_1$ formulas. At this point, it is not clear that there are any substructures of $\mathcal{N}$ in which this class of formulas behaves componentwise. In fact we will see from the following generalization of Theorem 2 that there substructures of $\mathcal{N}$ in which truth and componentwise truth coincide for formulas of any prescribed complexity. We will then be able to construct models of the $Th(\mathbb{N})$ in which all first order formulas behave componentwise.

Theorem 16 *Suppose M is a countable LA structure satisfying the Π_{n+1}-$Th(\mathbb{N})$ ($n \geq 0$). Then there is an embedding h of M into $\mathcal{N}$ such that in the structure $h(M)$ both the Σ_n and the Π_n formulas behave componentwise.*

Proof. The proof proceeds as in Theorem 2, except that instead of enumerating just Diophantine formulas, one expands the enumeration of polynomials to include Σ_n and Π_n formulas.

□

Theorem 17 *There are substructures of $\mathcal{N}$ satisfying $Th(\mathbb{N})$ for which truth and componentwise truth coincide.*

Proof. Let M be a countable model of $Th(\mathbb{N})$. Embed M in $\mathcal{N}$ as in Theorem 16, letting the ϕ_k's run through all first order formulas holding at $m_1, m_2, \ldots$. The argument of Theorem 16 shows that $h(M)$ is the required substructure of $\mathcal{N}$.

□

It follows that there are substructures of $\mathcal{N}$ in which the class of $\exists_1$ formulas behaves componentwise: Pick an arbitrary countable model M of the Π_2-$Th(\mathbb{N})$, and use Theorem 16 to construct an isomorphic copy in $\mathcal{N}$. In order to give some properties of these substructures, we need

Definition 18 *Let C be a class of L-structures, where L is a first order language. A structure M in C is said to be existentially closed (with respect to C) if any existential formula with parameters in M that holds in some extension of M belonging to C already holds in M. We shall be concerned only with the case that L is LA, and C is all models of $\forall_1$-$Th(\mathbb{N})$. We will then refer to M as simply e.c.*

The following proposition appears in [3]:

Proposition 19 *Every e.c. structure M satisfies Π_2-$Th(\mathbb{N})$.*

Proof. One first shows that every e.c. structure M satisfies the $\forall_2$-$Th(\mathbb{N})$. To this end, let $\Psi = \forall x \exists \vec{y} \phi(\vec{x}, \vec{y})$ be an $\forall_2$ sentence which holds in $\mathbb{N}$. Since the model M is e.c. it satisfies the $\forall_1$-$Th(\mathbb{N})$. Hence M extends to a model M' of $Th(\mathbb{N})$. (See [8]). The model M' satisfies Ψ, since $\mathbb{N}$ does. Now let $\vec{a}$ be a tuple from M. It suffices to show that $M \models \exists \vec{y} \phi(\vec{a}, \vec{y})$. But the structure M' satisfies $\exists \vec{y} \phi(\vec{a}, \vec{y})$. Therefore, since M is e.c., M must also satisfy $\exists \vec{y} \phi(\vec{a}, \vec{y})$. But $\vec{a}$ was arbitrary. Therefore $M \models \forall \vec{x} \exists \vec{y} \phi(\vec{x}, \vec{y})$, as required. Now the Π_2-$Th(\mathbb{N})$ is an inductive theory (see [2]). The Chang-Łoś-Suszko theorem states that inductive theories are $\forall_2$ axiomatizable. (See [1].) But the $\forall_2$ sentences belonging to the Π_2-$Th(\mathbb{N})$ are precisely the $\forall_2$-$Th(\mathbb{N})$. It follows that the $\forall_2$-$Th(\mathbb{N})$ proves the Π_2-$Th(\mathbb{N})$.

□

Proposition 20 *Suppose M is an e.c. substructure of $\mathcal{N}$. Then the class of $\exists_1$ formulas behaves componentwise in M.*

Proof. Let ϕ be an $\exists_1$ formula defined in M, and suppose $M \models \phi$. By Proposition 19, $M \models PA^-$. Thus, by Proposition 9, ϕ holds componentwise. Conversely, suppose ϕ holds componentwise. Let $\mathcal{G}$ be a non-principal ultrafilter in the boolean algebra of subsets of $\mathbb{N}$, and let $\mathbb{N}^\omega/\mathcal{G}$ be the ultrapower determined by $\mathcal{G}$. The LA structure $\mathbb{N}^\omega/\mathcal{G}$ is a model of $Th(\mathbb{N})$. Now there is a natural homomorphism $h : \mathcal{N} \longrightarrow \mathbb{N}^\omega/\mathcal{G}$ given by

$$[f] \mapsto \langle f \rangle,$$

where $\langle f \rangle$ is the class of the function f in $\mathbb{N}^\omega/\mathcal{G}$. The mapping h is injective on M: This is a consequence of the non-principality of $\mathcal{G}$, and the fact that if the formula $[f] \neq [g]$ holds in M, then it must hold componentwise. Let ϕ have the form $\exists x \psi(x, g)$, where g is in $\mathcal{N}$. (For simplicity, we assume ϕ has one quantifier and one parameter, but our argument is perfectly general.) Since ϕ holds componentwise, there is a function k from $\mathbb{N}$ to $\mathbb{N}$ such that $\psi(k, g)$ holds componentwise. Again because $\mathcal{G}$ is non-principal, it follows that $\psi(\langle k \rangle, \langle g \rangle)$, and hence $\exists x \psi(x, \langle g \rangle)$ hold in $\mathbb{N}^\omega/\mathcal{G}$. But $h(M)$ is e.c. (since h is injective), hence $\exists x \psi(x, \langle g \rangle)$ holds in $h(M)$. Thus ϕ holds in M, as required.

□

Remark 21 *The converse of Proposition 20 is false: There are substructures of $\mathcal{N}$ which are not existentially closed, but in which the class of $\exists_1$ formulas behaves componentwise. In fact, let M be any countable nonstandard model of $Th(\mathbb{N})$. By enumerating all formulas holding in $\mathbb{N}$ instead of just Diophantine formulas, we can find an isomorphic copy M' of M in $\mathcal{N}$ in which all formulas behave componentwise. But by a theorem of Rabin (Theorem 24 of this section), no nonstandard model of $Th(\mathbb{N})$ is e.c.*

Remark 22 *The embedding used in Proposition 20 is injective on every substructure of $\mathcal{N}$ satisfying the $\forall_1$-$Th(\mathbb{N})$. Since the ultrafilter $\mathcal{G}$ used there was* *arbitrary, it follows that every countable model of the* $\forall_1$-$Th(\mathbb{N})$ *can be embedded in every ultrapower* $\mathbb{N}^\omega/\mathcal{G}$. *This also follows from the* ω_1*-saturation of* $\mathbb{N}^\omega/\mathcal{G}$.

Our next result gives models of $Th(\mathbb{N})$ in which the $\exists_1$ formulas fail to behave componentwise.

Proposition 23 *Every countable nonstandard model* M *of* $Th(\mathbb{N})$ *has an isomorphic copy in* $\mathcal{N}$ *under which some* $\exists_1$ *formula fails to behave componentwise.*[5]

Proof. Let M be a countable nonstandard model of $Th(\mathbb{N})$. Let S be a simple subset of $\mathbb{N}$. This means S is an recursively enumerable set whose complement is infinite and contains no infinite recursively enumerable set. By the MRDP Theorem, there is an $\exists_1$ formula $\phi(x)$ defining S over $\mathbb{N}$. The sentence $\forall x \exists y(y > x \wedge \sim\phi(x))$ holds in $\mathbb{N}$, hence in M. Therefore there is a nonstandard element m of M such that $M \models \sim\phi(m)$. Let $m = m_1, m_2, \ldots$ be the distinct elements of M. We embed M in $\mathcal{N}$ as in Theorem 2. In the notation of that theorem, for all n, it is possible to choose $v_1(n)$ in S. Why? For each n, the formula in the single variable x_1, given by

$$\exists x_2, x_3, \ldots (P_1 \bigwedge \cdots \bigwedge P_n)$$

defines over $\mathbb{N}$ a recursively enumerable set S_n. Moreover, S_n is infinite: Since m_1 is nonstandard, there are P_k's of the form $x_1 = x_i + r$ for arbitrarily large integers r. It follows from the fact that S is simple that each S_n meets S, so we can choose $v_1(n)$ in S for all n, as asserted. But then we obtain an embedding h such that the $\exists_1$ formula $\phi(h(m_1))$ holds componentwise. Since $\phi(m)$ fails to hold in M, and since h is an embedding, the formula $\phi(h(m_1))$ does not hold in $h(M)$. So $\phi(h(m_1))$ does not behave componentwise in $h(M)$.

□

Propositions 20 and 23 imply:

Theorem 24 (Rabin, 1962) [6] *If* M *is a nonstandard model of* $Th(\mathbb{N})$ *then* M *is not e.c.*

[5] M need only be a model of the Σ_2-theory of $\mathbb{N}$.
[6] Rabin's 1962 proof did not make use of the MRDP Theorem. See [10].

Proof. Assume, on the contrary, that M is e.c. Then any countable elementary substructure of M is also e.c. (This is a consequence of the joint embedding property for models of the $\forall_1$-$Th(\mathbb{N})$. (See [5].) Thus we can assume that M is countable. By Proposition 23, M has an isomorphic copy in $\mathcal{N}$ in which some $\exists_1$ formula fails to behave componentwise. But by Proposition 20, the $\exists_1$ formulas behave componentwise in every countable e.c. substructure of $\mathcal{N}$, a contradiction.

□

If M is a substructure of $\mathcal{N}$, and if the $\exists_1$ formulas behave componentwise in M, what theory must M satisfy?

Proposition 25 *If the $\exists_1$ formulas behave componentwise in the substructure M of $\mathcal{N}$, then $M \models \Pi_2$-$Th(\mathbb{N})$. Conversely, if $M \models \Pi_2$-$Th(\mathbb{N})$, then the $\exists_1$ formulas behave componentwise in some isomorphic copy of M in $\mathcal{N}$.*

Proof. Assume the $\exists_1$ formulas behave componentwise in M. First, we shall show that $M \models \forall_2$-$Th(\mathbb{N})$. Let $\phi(\vec{x}, \vec{y})$ be an open formula, and suppose $\mathbb{N} \models \forall\vec{x}\exists\vec{y}\phi(\vec{x}, \vec{y})$. Let $\vec{f}$ be a tuple from M. Then the formula $\exists\vec{y}\phi(\vec{f}, \vec{y})$ holds componentwise, hence it holds in M. Hence $M \models \forall\vec{x}\exists\vec{y}\phi(\vec{x}, \vec{y})$, as required. But it is known that $\forall_2$-$Th(\mathbb{N})$ proves the Π_2-$Th(\mathbb{N})$. The second assertion follows from Theorem 16.

□

Finally, we note that if M is a substructure of $\mathcal{N}$, and if, in M, the $\exists_1$ formulas behave componentwise, then so must the Σ_1 formulas: For by the previous proposition, $M \models \Pi_2$-$Th(\mathbb{N})$. Each Σ_1 formula is equivalent to an $\exists_1$ formula over $\mathbb{N}$, and this equivalence persists in M, since it is itself Π_2. Thus these two classes of formulas behave componentwise in exactly the same substructures of $\mathcal{N}$.

6 Cohesiveness

In this section we answer the following question: Let T be one of the theories $\forall_1$-$Th(\mathbb{N})$, Π_2-$Th(\mathbb{N})$, $Th(\mathbb{N})$. For which functions f from $\mathbb{N}$ to $\mathbb{N}$ is there a model of T inside $\mathcal{N}$ which contains f? We will not answer this question for general LA theories T, but we will indicate the form that such an answer might take. We need a

Definition 26 *Let $\mathcal{L}$ be a subset of $\mathcal{P}(\mathbb{N})$, the power set of $\mathbb{N}$. Let f be a function from $\mathbb{N}$ to $\mathbb{N}$. We say that f is $\mathcal{L}$-cohesive if for all s in $\mathcal{L}$, the function f either eventually assumes values in s or eventually assumes values outside of s.*[7]

For example, if $\mathcal{L}$ is the set of all finite subsets of $\mathbb{N}$, then a function f is $\mathcal{L}$-cohesive iff it is eventually constant or tends to infinity. If $\mathcal{L}$ is the set of all recursive sets of integers, then an $\mathcal{L}$-cohesive function is eventually prime or eventually composite, eventually even or eventually odd, and so on. In this case we will refer to f as r-cohesive. In the the case that $\mathcal{L}$ is the set of recursively enumerable sets of integers, we will refer to f as simply cohesive.[8]

As a first application of this definition, we establish

Proposition 27 *Let $\mathcal{L}$ be the set of finite subsets of $\mathbb{N}$. A function f is contained in a substructure of $\mathcal{N}$ satisfying $\forall_1$-$Th(\mathbb{N})$ iff f is $\mathcal{L}$-cohesive.*

Proof. Suppose f is $\mathcal{L}$-cohesive. Either f is eventually constant or f tends to infinity. In the first case, the substructure of $\mathcal{N}$ generated by f is isomorphic to the standard model, which satisfies the required theory. Assume f tends to infinity. Then the substructure generated by f is isomorphic to the polynomial semiring $\mathbb{N}[x]$. But this is a model of $\forall_1$-$Th(\mathbb{N})$, see remarks after Proposition 9.

Conversely, assume f is contained in a model of $\forall_1$-$Th(\mathbb{N})$. In the discussion following Theorem 2, we show that f is either eventually constant or tends to infinity.

□

The next sequence of lemmas is devoted to proving an analog of Proposition 27 for Π_2-$Th(\mathbb{N})$.

Lemma 28 *Let $M \subseteq \mathcal{N}$ be a model of the Π_2-$Th(\mathbb{N})$. If $[f] \in M$, then the function f is r-cohesive.*

Proof. Let R be a recursive set of natural numbers, and let S be the complement of R in $\mathbb{N}$. Making use of the MRDP Theorem, let $\rho(x)$ and $\sigma(x)$ be $\exists_1$ formulas defining R and S in the structure $\mathbb{N}$. We note that M satisfies

[7] There is an analogous definition of cohesive sets of integers in [11], page 231. A function f tending to infinity is cohesive in our sense iff it has a cohesive range in Rogers' sense.

[8] This conforms to Rogers' definitions for sets.

the sentence $\forall x(\rho(x) \vee \sigma(x))$, since this sentence holds in $\mathbb{N}$, and is logically equivalent to a Π_2 sentence. Thus M satisfies one of the formulas $\rho(f)$, $\sigma(f)$. In the first case, by Proposition 9, $\rho(f)$ holds componentwise. In the second case $\sigma(f)$ holds componentwise. But this means either f is eventually in R or f is eventually in S. Thus f is r-cohesive.

□

Lemma 29 *Let M be a substructure of $\mathcal{N}$ which satisfies the $\forall_1$-$Th(\mathbb{N})$. Suppose further that M is closed under the componentwise application of recursive functions, i.e., given any recursive function $g : \mathbb{N} \longrightarrow \mathbb{N}$, if $[f] \in M$ then $[g \circ f] \in M$. If $[f]$ is an element of M, then f is r-cohesive.*

Proof. Let R be a recursive set of natural numbers with characteristic function χ_R. We observe that $[\chi_R \circ f] \in M$, since M is closed under the componentwise application of recursive functions. Since M satisfies $\forall_1$-$Th(\mathbb{N})$, it follows from Proposition 27 that the function $\chi_R \circ f$ is $\mathcal{L}$-cohesive, where $\mathcal{L}$ is the set of finite subsets of $\mathbb{N}$. By the discussion preceding Proposition 27, $\chi \circ f$ is eventually constant. Hence f is eventually in R or eventually in the complement of R, as required.

□

Cohesiveness is related in a natural way to componentwise behavior:

Lemma 30 *Let M be a substructure of $\mathcal{N}$ which satisfies the $\forall_1$-$Th(\mathbb{N})$. Suppose that M is closed under the componentwise application of recursive functions from $\mathbb{N}$ to $\mathbb{N}$. Then the Δ_0 formulas behave componentwise in M.*

Proof. We note first that if $g : \mathbb{N}^m \to \mathbb{N}$ is recursive (for any $m \geq 0$) and $f_1, \ldots, f_m$ are in M, then so is $g(f_1, \ldots, f_m)$. To see this, let $\sigma_n(x_1, \ldots, x_n)$ be the iterated Cantor pairing function, defined inductively as follows.

$$\begin{aligned}
&\sigma_1(x) = x \\
&\sigma_2(x, y) = \tfrac{(x+y)(x+y+1)}{2} + y \\
&\sigma_n(x_1, \ldots, x_n) = \ \sigma_2(x_1, \sigma_{n-1}(x_2, \ldots x_n)) \quad \text{for } \ n \geq 2
\end{aligned}$$

The function $2^m \cdot \sigma_m$ is a polynomial σ'_m over $\mathbb{N}$. It follows that the function $\sigma'_m(f_1, \ldots, f_m)$ is in M. Let $g' : \mathbb{N} \to \mathbb{N}$ be defined by

$$g'(x) = \begin{cases} g(x_1, \ldots, x_m) & \text{if } x = \sigma'_m(x_1, \ldots, x_m) \\ 0 & \text{if } x \text{ is not in the range of } \sigma'_m \end{cases}$$

Then the function $g' \circ \sigma'_m(f_1, \ldots, f_m)$ is in M, i.e., $g(f_1, \ldots, f_m)$ is in M, as required.

We now turn to the proof that Δ_0 formulas behave componentwise in M. We proceed by induction on the complexity of the Δ_0 formula ϕ.

If ϕ is atomic then it is immediate that ϕ behaves componentwise. If ϕ is a conjunction then the componentwise behavior of ϕ follows immediately by induction.

Assume now that ϕ has the form $\sim\psi(\vec{x})$. For this case we will require the Cantor projection functions $\pi_i(x)$, which we define as follows:

$$y = \pi_i(x) \quad \text{iff} \quad \exists w_1, \ldots, w_{i+1} \leq x (x = \sigma_{i+1}(w_1, \ldots, w_{i+1}) \wedge y = w_i).$$

The π_i are total recursive functions from $\mathbb{N}$ to $\mathbb{N}$ satisfying

$$\pi_i(\sigma_{i+1}(x_1, \ldots, x_{i+1})) = x_i.$$

Thus, by the closure property of M, if $[f]$ is in M then so is $[\pi_i \circ f]$.

Now suppose $M \models\sim \psi(f_1, \ldots, f_k)$. Then it is not the case that $M \models \psi(f_1, \ldots, f_k)$. Therefore by

the induction hypothesis it is not the case that $\psi(f_1, \ldots, f_k)$ holds componentwise. This means, for infinitely many n,

$$\mathbb{N} \models\sim\psi(f_1(n), \ldots, f_k(n)).$$

By the closure property of M, the function $f(x)$ defined by $\sigma_n(f_1(x), \ldots, f_n(x))$ is an element of M. Hence f is r-cohesive, and $\pi_i(f(x)) = f_i(x)$ for all $i \leq n$. But the formula $\sim\psi(\pi_1(x), \ldots, \pi_k(x))$ defines a recursive subset of $\mathbb{N}$ containing $f(n)$ for infinitely many n. It follows that f is eventually in this set. Hence, the formula $\sim\psi(f_1, \ldots, f_k)$ holds componentwise, as required.

Conversely, suppose that $\sim\psi(f_1, \ldots, f_k)$ holds componentwise. Then it is not the case that the formula $\psi(f_1, \ldots, f_k)$ holds componentwise. Thus, by induction, the formula $\sim\psi(f_1, \ldots, f_k)$ must hold in M.

Finally, suppose ϕ has the form $\exists x \leq y\ \psi(x, y, z_1, \ldots, z_m)$, and suppose that

$$M \models \exists x \leq f\ \psi(x, f, g_1, \ldots, g_m).$$

Then

$$M \models \psi(h, f, g_1, \ldots, g_m) \wedge h \leq f$$

for some $h \in M$. By the induction hypothesis and the componentwise behavior of atomic formulas, the formula

$$\psi(h, f, g_1, \ldots, g_m) \wedge h \leq f$$

holds componentwise. But this means the formula

$$\exists x \leq f \ \psi(x, f, g_1, \ldots, g_m)$$

holds componentwise.

Conversely suppose the formula $\exists x \leq f \ \psi(x, f, g_1, \ldots, g_m)$ holds componentwise. We wish to show that

$$M \models \exists x \leq f \ \psi(x, f, g_1, \ldots, g_m).$$

Define $r : \mathbb{N}^{m+1} \to \mathbb{N}$ as follows: $r(u, v_1, \ldots, v_m)$ is the least x such that $\mathbb{N} \models x \leq u \wedge \psi(x, u, v_1, \ldots, v_m)$, if there is such an x, and 0 otherwise. The function r is total recursive. Let s be defined by

$$s(n) = r(f(n), g_1(n). \ldots, g_m(n)).$$

Then $[s]$ is in M, by the closure property of M and the note at the beginning of this lemma. But we have defined s so that the formula $\psi(s, f, g_1, \ldots, g_m) \wedge s \leq f$ holds componentwise. Thus, by induction,

$$M \models \psi(s, f, g_1, \ldots, g_m) \wedge s \leq f.$$

Therefore

$$M \models \exists x \leq f \psi(x, f, g_1, \ldots, g_m),$$

as required.

□

We mention the following characterization of models of $\Pi_2 - Th(\mathbb{N})$:

Theorem 31 (Kennedy-Raffer) *Let M be a countable substructure of $\mathcal{N}$ satisfying the $\forall_1$-$Th(\mathbb{N})$. Then $M \models \Pi_2$-$Th(\mathbb{N})$ iff M is closed under the compoponentwise application of total recursive functions.*

Proof. Suppose $M \models \Pi_2$-$Th(\mathbb{N})$ and suppose $g : \mathbb{N} \to \mathbb{N}$ is total recursive. Let f be an element of M. We wish to show that $g \circ f$ is in M. Using the MRDP

Theorem, we choose an $\exists_1$ formula $\delta(x,z)$ defining the graph of g in $\mathbb{N}$. That is, for all x and y in $\mathbb{N}$,

$$\mathbb{N} \models \delta(x,y) \text{ iff } y = g(x).$$

The Π_2 formula $\forall x \exists! y \delta(x,y)$ holds in $\mathbb{N}$, hence it also holds in M. Choose a function h in M such that $M \models \delta(f,h)$. Since M satisfies $\forall_1$-$Th(\mathbb{N})$, by Proposition 9, the formula $\delta(f,h)$ holds componentwise. Also the formula $\delta(f, g \circ f)$ holds componentwise. Since δ defines in $\mathbb{N}$ the graph of a function, it follows that $[h] = [g \circ f]$. Thus $[g \circ f]$ is in M, as required.

Conversely, suppose M has the stated closure property. Let $\phi(\vec{x}, \vec{y})$ be a Δ_0 formula and let Φ be the sentence $\forall \vec{x} \exists \vec{y} \phi(\vec{x}, \vec{y})$. We assume that Φ holds in $\mathbb{N}$, and we wish to show that it holds in M. Choose $f_1, \ldots, f_m$ arbitrarily from M. It suffices to show that $M \models \exists \vec{y} \phi(f_1, \ldots, f_m, \vec{y})$. To this end, we define the function h on $\mathbb{N}^m$ by:

$$h(\vec{x}) = \text{the least } y \text{ such that } \mathbb{N} \models \phi(\vec{x}, \pi_1(y), \ldots, \pi_n(y)),$$

where the functions π_i are defined in Lemma 30. Because $\mathbb{N}$ satisfies Φ, h is a total function. Moreover h is recursive. Now let f be the function defined on $\mathbb{N}$ by

$$f(x) = h(f_1(x). \ldots, f_m(x)).$$

By the remarks at the beginning of the proof of Lemma 30, the function f must be in M. Thus, by the assumed closure property of M, the functions $\pi_1 \circ f, \ldots, \pi_n \circ f$ are all in M. But, since Φ holds in $\mathbb{N}$, the formula

$$\phi(f_1, \ldots, f_m, \pi_1 \circ f, \ldots, \pi_n \circ f)$$

holds componentwise. By Lemma 30, this formula holds in M. Therefore $M \models \exists \vec{y} \phi(f_1, \ldots, f_m, \vec{y})$, and the proof is complete.

□

We can now answer the question, concerning Π_2-$Th(\mathbb{N})$, stated at the beginning of this section:

Theorem 32 *Let f be a function from $\mathbb{N}$ to $\mathbb{N}$. Then f is contained in some substructure of $\mathcal{N}$ satisfying Π_2-$Th(\mathbb{N})$ iff f is r-cohesive.*

Proof. Suppose $M \subseteq \mathcal{N}$ is a model of Π_2-$Th(\mathbb{N})$ containing f. By Lemma 28, f is r-cohesive.

Conversely, suppose f is r-cohesive. Let

$$M = \{[g \circ f] : g \text{ is a recursive function from } \mathbb{N} \text{ to } \mathbb{N}\}.$$

We will show that M is a model of Π_2-$Th(\mathbb{N})$. Since the recursive functions are closed under addition and multiplication, M is a substructure of $\mathcal{N}$. Moreover, M is closed under the componentwise application of total recursive functions. We wish to apply Theorem 31, but first we have to prove that M satisfies $\forall_1$-$Th(\mathbb{N})$. By Theorem 12, it suffices to show that M is a model of PA^-. Now M inherits from $\mathcal{N}$ the axioms for a partially ordered semiring, since these axioms are $\forall_1$. Thus we need only check that the order relation in M is total and discrete, and that M is closed under nonnegative differences. For totality, suppose g and h are recursive functions, and suppose $M \models \sim [g \circ f] \leq [h \circ f]$. Then f does not eventually assume values in the set $\{n : g(n) \leq h(n)\}$. But f is r-cohesive, and this is a recursive set. Thus f eventually assumes values in $\{n : h(n) \leq g(n)\}$. Thus $M \models [h \circ f] \leq [g \circ f]$. This proves that $\leq$ is total in M. Discreteness (which asserts that if $x \leq y \leq x + 1$ then $x = y$ or $x = y + 1$) is proved similarly. As for nonnegative differences, we observe that the function giving the nonnegative difference of two integers is recursive, thus by an argument similar to the one at the beginning of Lemma 30, M is closed under this function. We have shown that M is a model of PA^-. It follows that M satisfies $\forall_1$-$Th(\mathbb{N})$. Thus, by Theorem 31, M satisfies Π_2-$Th(\mathbb{N})$.

□

Next, we consider the problem, when is a function f contained in a model of $Th(\mathbb{N})$? Cohesiveness provides a sufficient condition:

Theorem 33 (Kennedy-Raffer) *Let f be cohesive. Then f is contained in a substructure $M \subseteq \mathcal{N}$ satisfying $Th(\mathbb{N})$.*

Proof. Let $\phi_1(x), \phi_2(x), \ldots$ be all $\exists_1$ and $\forall_1$ formulas in one free variable such that for each i, f is eventually in $\phi_i^{\mathbb{N}}$. (If $\phi(x)$ is a formula and M is a structure, we use ϕ^M to denote $\{m \in M : M \models \phi(m)\}$.) Let c be a new constant symbol added to LA, the language of arithmetic, and let Γ be the theory

$$Th(\mathbb{N}) \cup \{\phi_1(c), \phi_2(c), \ldots\}.$$

Because f is eventually in each of the sets $\phi_i^{\mathbb{N}}$, we can satisfy any finite subset of Γ in the standard model $\mathbb{N}$, by taking $c^{\mathbb{N}}$ sufficiently large.

Choose, by compactness, a countable model M of Γ. To prove the theorem, we will construct an embedding of M into $\mathcal{N}$ which maps the element c^M to $[f]$.

Let $m_1, m_2, \ldots$ be an enumeration of M for which $c^M = m_1$. Let $P_1, P_2, \ldots$ be a sequence of polynomials which can be used to construct an embedding of M into $\mathcal{N}$, as in Theorem 2. That is, for each n,

$$M \models P_n(x_1/m_1, x_2/m_2, \ldots).$$

Let $D_n(x_1)$ be the formula $\exists x_2, x_3, \ldots P_n(x_1, x_2, \ldots, x_k)$. It would be possible to map c^M to $[f]$ using the argument of Theorem 2, if we knew that $\mathbb{N} \models D_n(f(n))$ for all sufficiently large n. However, the P_n's may be ordered in such a way that this is impossible. We will show that there is a re-ordering of the P_n's such that an embedding of the required type can be constructed.

We first observe that for each k, f is eventually in $D_k^{\mathbb{N}}$. To see this, suppose otherwise. Then because the function f is cohesive, and because the set $D_k^{\mathbb{N}}$ is recursively enumerable, it must be that f is eventually in $\sim D_k^{\mathbb{N}}$. Thus $\sim D_k$ appears as one of the $\phi's$, say ϕ_i. But for all n, $M \models \phi_n(m_1)$. Thus $M \models \sim D_i(m_1)$, a contradiction.

Now for each integer i, let $P_{i1}, P_{i2}, \ldots$ be an enumeration of the following set of polynomials:

$$\{P_j : j \geq 1 \text{ and } \text{ for all } n \geq i, f(n) \in D_j^{\mathbb{N}}\}.$$

We have shown that f eventually assumes its values in each of the sets $D_n^{\mathbb{N}}$. It follows that each of the P_n's appears (in fact infinitely often) as one of the P_{ij}'s. Now let

$$Q_n = \bigwedge_{0 \leq i,j, \leq n} P_{ij}(x_1, \ldots, x_n).$$

It follows that for each n,

$$\mathbb{N} \models \exists x_2, x_3, \ldots Q_n(f(n), x_2, x_3, \ldots).$$

Finally, choose, at each stage n, a sequence of natural numbers $v_2(n), v_3(n), \ldots$ such that

$$\mathbb{N} \models Q_1 \bigwedge \ldots \bigwedge Q_n(x_1/f(n), x_2/v_2(n), \ldots).$$

With the Q_n's replacing the P_n's, we then obtain the embedding we set out to construct, i.e., an embedding which sends c^M to $[f]$.

□

We do not know a necessary and sufficient condition for a function f to be contained in a substructure of $\mathcal{N}$ satisfying $Th(\mathbb{N})$. We now give an example to show that the condition given in Theorem 33 is not necessary.

Proposition 34 *There is a substructure M of $\mathcal{N}$ satisfying the $Th(\mathbb{N})$, and a function f in M such that f is not cohesive.*

Proof. Let M be a countable nonstandard model of $Th(\mathbb{N})$. We will show that there is an embedding h of M into $\mathcal{N}$ such that under h, an element of M is mapped to a function which is not cohesive. Let the $\exists_1$ formula $\phi(x)$ define a simple set S in $\mathbb{N}$. Arguing as in Proposition 23, there is a nonstandard element $m \in M$ such that $M \models \sim \phi(m)$. Using the notation of Theorem 2 and Theorem 4, let $P_1, P_2, \ldots$ be all the polynomial equations over $\mathbb{N}$ such that $M \models P_i(x_1/m_1, x_2/m_2, \ldots)$. We construct our embedding as follows: At stage n, for n odd, choose natural numbers $v_1(n), v_2(n), \ldots$ for which

$$\mathbb{N} \models P_1 \bigwedge \cdots \bigwedge P_n(x_1/v_1(n), x_2/v_2(n), \ldots,) \wedge \phi(v_1(n)).$$

That such a tuple of natural numbers exists follows from the simplicity of $\phi^{\mathbb{N}}$, as in the proof of Proposition 23. At stage n, for n even, choose natural numbers $v_1(n), v_2(n), \ldots$ for which

$$\mathbb{N} \models P_1 \bigwedge \cdots \bigwedge P_n(x_1/v_1(n), x_2/v_2(n), \ldots,) \wedge \sim \phi(v_1(n)).$$

That such a tuple of natural numbers exists follows from the fact that we can expand the set of formulas witnessed in the construction of the embedding as in Theorem 16. As in Theorem 2 and Theorem 4, we define our embedding $h : M \to \mathcal{N}$ by:

$$m_i \longmapsto [\lambda k.v(i(k)].$$

But then we obtain an embedding h such that $v_1(n) \in S$, if n is even, $v_1(n) \notin S$, if n is odd. But this means the function $v_1(n)$ is not cohesive.

□

References

[1] Chang, C. C.; Keisler, H. J., *Model Theory*, Amsterdam: North Holland, (1973).

[2] Hirschfeld, J., *Models of arithmetic and recursive functions*, Israel Journal of Mathematics #20 (1975), 111–126.

[3] Hirschfeld, J., *Forcing, Arithmetic and Division Rings*, Springer Lecture Notes in Mathematics, vol. 454.

[4] Hodges, W., *Model Theory*, Cambridge, U.K.: Cambridge University Press, (1993).

[5] ———, *Building Models by Games*, London Mathematical Society Student Texts:2, Cambridge, U.K.: Cambridge University Press.

[6] Kaye, R., *Open induction, Tennenbaum phenomena, and complexity theory*, In: Arithmetic, Proof Theory and Computational Complexity, J. Krajíček P. Clote, ed., Oxford, U.K., Oxford University Press, (1933), 222-237.

[7] ———, *Models of Peano Arithmetic*, Oxford Logic Guides, Oxford: Oxford University Press, (1991).

[8] Kennedy, J., *On Embedding Models of Arithmetic into Reduced Powers*, Ph.D Dissertation, The Graduate School and University Center of the City University of New York, (1996).

[9] Lang, S., *Algebra*, Reading, Massachusetts: Addison Wesley, first edn., (1971).

[10] Rabin, M., *Diophantine equations and nonstandard models of arithmetic*, In: Logic, Methodology and Philosophy of Science, A. Tarski E. Nagel, P. Suppes, ed. Stanford, (1962).

[11] Rogers, H., *Theory of Recursive Functions and Effective Computability*, Cambridge, Massachusetts: MIT Press, second edn., (1987).

[12] Thoralf Skolem, *Peano's axioms and models of arithmetic*, In: Mathematical Interpretations of Formal Systems, Amsterdam, (1955), North Holland.

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