QUANTITATIVE ANALYSIS OF METASTABILITY IN REVERSIBLE DIFFUSION PROCESSES VIA A WITTEN COMPLEX APPROACH

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1 Introduction

We are interested in the exponentially small eigenvalues of the semiclassical Witten Laplacian on 0-forms

$$\Delta_{f,h}^{(0)} = -h^2\Delta + |\nabla f(x)|^2 - h\Delta f(x).$$

We shall consider this operator on Ω which is either a connected compact Riemannian manifold or $\mathbb{R}^n$. The function f will be a Morse function and when Ω is a compact manifold for example it is known (see [Wit], [CFKS] and [HelSj3]) that there are exactly m_0 eigenvalues in some interval $[0, e^{-\alpha/h}]$ for $h > 0$ small enough, where m_0 is the number of local minima. Moreover the same result holds for Witten Laplacians on p-forms if m_p denotes the number of critical points of index p.

Our purpose is to derive accurate asymptotic formulas for the m_0 first eigenvalues of $\Delta_{f,h}^{(0)}$. A similar problem was considered by many authors via a probabilistic approach in [HolKusStr], [Mi], [Ko], and more recently in [BEGK] and [BGKl], where A. Bovier, V. Gayrard and M. Klein obtained accurate asymptotic forms of the exponentially small eigenvalues. The Witten Laplacian being associated to the Dirichlet form

$$u \mapsto \int_\Omega |h\nabla u(x)|^2 \; e^{-2f(x)/h} \; dx,$$

they considered this problem via a probabilistic approach. They obtained the following asymptotic behaviour for the first nonzero eigenvalues $\lambda_k(h)$, $k \in \{2, \dots, m_0\}$, with $\lambda_1(h) = 0$ [1], of $\Delta^{(0)}_{f,h}$:

$$\lambda_k(h) = \frac{h}{\pi} |\widehat{\lambda}_1(U^{(1)}_{j(k)})| \sqrt{\frac{\left|\det(\operatorname{Hess} f(U^{(0)}_k))\right|}{\left|\det(\operatorname{Hess} f(U^{(1)}_{j(k)}))\right|}} \times \exp -\frac{2}{h}\left(f(U^{(1)}_{j(k)}) - f(U^{(0)}_k)\right) \times (1 + \mathcal{O}(h^{\frac{1}{2}} |\ln h|)) \,, \quad (1.1)$$

where the $U^{(0)}_k$ denote the local minima of f ordered in some specific way, the $U^{(1)}_{j(k)}$ are "saddle points" attached in a specific way to the $U^{(0)}_k$ (which appear to be critical points of index 1) and $\widehat{\lambda}_1(U^{(1)}_{j(k)})$ is the negative eigenvalue of $\operatorname{Hess} f(U^{(1)}_{j(k)})$.
Beside the fact that one would like to relate this result to the previous semiclassical analysis by Helffer-Sjöstrand of the Witten complex in [HelSj3], our aim is twofold :

1) Improve the remainder and replace the $\mathcal{O}(h^{1/2} \ln h)$-term by $\mathcal{O}(h)$ with a possible higher order expansion.

2) Extend the results of Bovier-Gayrard-Klein to the cases when Ω is an oriented Riemannian manifold or when $\Omega = \mathbb{R}^n$ and $e^{-f(x)/h}$ does not belong to L^2, which cannot be handled easily via the probabilistic approach.

Although the present approach leads to more accurate and general results, the probabilistic point of view presents other interests :

a) First of all, the probabilistic interpretation and its link with potential theory gave to these authors the right intuition for the geometrical quantities involved in the asymptotic behaviour of the exponentially small eigenvalues. Indeed the numbering of local minima and the choice of the critical point of index 1, $U^{(1)}_{j(k)}$, associated with $U^{(0)}_k$, is given by ordering the exit

[1] Our analysis will permit to consider cases when $e^{-\frac{2f}{h}} \notin L^1(\Omega)$. For such cases (1.1) also holds for $k = 1$.

times from a valley for the stochastic process associated with the Dirichlet form. Moreover the quantities involved in (1.1) can be expressed in terms of capacities.

b) Their method requires only $f \in \mathcal{C}^3(\Omega)$, while our analysis, although it could be carried out with low regularity assumptions, is more efficiently presented with $f \in \mathcal{C}^\infty(\Omega)$.

Although it will require some estimates and constructions present in the WKB analysis of Helffer-Sjöstrand in [HelSj3], our approach will follow a slightly different strategy. We will use more extensively the complex structure of the Witten Laplacian and the fact that we are looking at $\Delta^{(0)}_{f,h}$. We recall that

$$\Delta_{f,h} = d_{f,h} d^*_{f,h} + d^*_{f,h} d_{f,h} \ ,$$

where $d_{f,h}$ is the distorted differential $e^{-f(x)/h}\,(hd_x)\,e^{f(x)/h}$ and $d^*_{f,h}$ its adjoint for the Riemannian structure. The restriction of $d_{f,h}$ to p-forms is denoted by $d^{(p)}_{f,h}$ and we have

$$\Delta^{(0)}_{f,h} = d^{(0)*}_{f,h} d^{(0)}_{f,h}.$$

In the Witten-complex spirit, we will consider the singular values of the restricted differential $d^{(0)}_{f,h} \ : F^{(0)} \to F^{(1)}$, which will be more shortly denoted by $\beta^{(0)}_{f,h}$,

$$\beta^{(0)}_{f,h} := (d^{(0)}_{f,h})_{/F^{(0)}} \ , \tag{1.2}$$

where $F^{(\ell)}$ is the m_ℓ-dimensional spectral subspace of $\Delta^{(\ell)}_{f,h}$, $\ell \in \{0,1\}$,

$$F^{(\ell)} = \mathrm{Ran}\ 1_{[0,Ch^{3/2})}(\Delta^{(\ell)}_{f,h}) \ , \tag{1.3}$$

with the property

$$1_{[0,Ch^{3/2})}(\Delta^{(1)}_{f,h}) d^{(0)}_{f,h} = d^{(0)}_{f,h} 1_{[0,Ch^{3/2})}(\Delta^{(0)}_{f,h}) \ . \tag{1.4}$$

Because the value of $C > 0$ does not play any role (for h small enough), we will choose from now on $C = 1$. More generally one could define a complex $\beta^{(\ell)}_{f,h}$ by

restriction of $d^{(\ell)}_{f,h}$ to the $F^{(\ell)}$ but one will mainly concentrate on the cases $\ell = 0$ and $\ell = 1$.
Working with singular values of $\beta^{(0)}_{f,h}$ happens to be more efficient than considering their squares as the eigenvalues of $\Delta^{(0)}_{f,h}$, in order to exploit all the information which can be extracted from well chosen quasimodes.

Finally we mention that this problem was presented and treated in a particular case in [HelNi]. Application of quantitative accurate estimates for the first non zero eigenvalue of the Witten Laplacian in connection with the return to the equilibrium for the Fokker-Planck equation of kinetic theory can be found in [HerNi] and [HelNi].

This article (introduction excluded) is now divided in five sections. In Section 2, we specify our conditions on the function f in order to have self-adjoint Witten Laplacians with good spectral properties. In Section 3, we first specify the notion of "(strict) saddle point" in the different cases. After this we are in a position to write the main assumption which excludes degenerate eigenvalues. In Section 4, we introduce some specific cut-off functions and the corresponding quasimodes for $\Delta^{(0)}_{f,h}$ and $\Delta^{(1)}_{f,h}$. This is only in Section 5 (Theorem 5.1) that we state accurately our result by making use of the precise notions introduced before. Section 6 is devoted to the core of the proof of Theorem 5.1. It involves an induction process which makes an efficient use of the previous estimates on quasimodes.

2 Morse functions and Witten Laplacians.

2.1 Witten complexes and associated Laplacians

Let Ω be an n-dimensional connected compact oriented Riemannian manifold or $\mathbb{R}^n$. Depending on the cases $\overline{\Omega}$ will be Ω or $\mathbb{R}^n \sqcup \{\infty\}$. The cotangent (resp. tangent) bundle is denoted $T^*\Omega$ (resp. $T\Omega$) and the exterior fiber bundle $\Lambda T^*\Omega = \oplus_{p=0}^{n} \Lambda^p T^*\Omega$ (resp. $\Lambda T\Omega = \oplus_{p=0}^{n} \Lambda^p T\Omega$). The space of $\mathcal{C}^\infty$, $\mathcal{C}^\infty_0$,

L^2 ...sections in any of these fiber bundles, E, will be denoted respectively $\mathcal{C}^\infty(\Omega; E)$, $\mathcal{C}_0^\infty(\Omega; E)$, $L^2(\Omega; E)$.... When no confusion is possible we will simply use the short notations $\Lambda^p\mathcal{C}^\infty$, $\Lambda^p\mathcal{C}_0^\infty$ and $\Lambda^p L^2$ for $E = \Lambda^p$. The differential on $\mathcal{C}_0^\infty(\Omega; \Lambda T^*\Omega)$ will be denoted by d and more precisely

$$d^{(p)} : \mathcal{C}_0^\infty(\Omega; \Lambda^p T^*\Omega) \to \mathcal{C}_0^\infty(\Omega; \Lambda^{p-1} T^*\Omega).$$

Its formal adjoint with respect the L^2-scalar product inherited from the riemanian structure is denoted by d^* with

$$d^{(p),*} : \mathcal{C}_0^\infty(\Omega; \Lambda^{p+1} T^*\Omega) \to \mathcal{C}_0^\infty(\Omega; \Lambda^p T^*\Omega).$$

For a Morse function $f \in \mathcal{C}^\infty(\Omega; \mathbb{R})$ we set

$$d_{f,h} = e^{-f(x)/h}\,(hd)\,e^{f(x)/h} \quad \text{and} \quad d_{f,h}^* = e^{-f(x)/h}\,(hd)^*\,e^{f(x)/h}.$$

The Witten Laplacian is defined as

$$\Delta_{f,h} = d_{f,h}^* d_{f,h} + d_{f,h} d_{f,h}^*,$$

which means

$$\Delta_{f,h}^{(p)} = d_{f,h}^{(p),*} d_{f,h}^{(p)} + d_{f,h}^{(p-1)} d_{f,h}^{(p-1),*} : \mathcal{C}_0^\infty(\Omega; \Lambda^p T^*\Omega) \to \mathcal{C}_0^\infty(\Omega; \Lambda^p T^*\Omega).$$

Note that $d_{f,h} d_{f,h} = 0$ and $d_{f,h}^* d_{f,h}^* = 0$ respectively imply, that for all u in $\mathcal{C}_0^\infty(\Omega; \Lambda^p T^*\Omega)$,

$$\Delta_{f,h}^{(p+1)} d_{f,h}^{(p)} u = d_{f,h}^{(p)} \Delta_{f,h}^{(p)} u \tag{2.1}$$

and

$$\Delta_{f,h}^{(p-1)} d_{f,h}^{(p-1),*} u = d_{f,h}^{(p-1),*} \Delta_{f,h}^{(p)} u. \tag{2.2}$$

The next assumption leads to a good self-adjoint realization of $\Delta_{f,h}$ with similar basic properties in all cases.

Assumption 2.1. *The function f belongs to $\mathcal{C}^\infty(\Omega)$ and is a Morse function. Moreover, in the case when $\Omega = \mathbb{R}^n$, there is a compact set $K \subset \mathbb{R}^n$ and a constant $C > 0$ such that*

$$\forall x \in \mathbb{R}^n \setminus K, \qquad |\nabla f(x)| \geq \frac{1}{C} \tag{2.3}$$

$$\forall x \in \mathbb{R}^n \setminus K, \qquad |\mathrm{Hess}\, f(x)| \leq C |\nabla f(x)|^2 \,. \tag{2.4}$$

With inequality (2.3), the Morse function f has only a finite number of critical points in Ω. The set of all critical points of index p will be called $\mathcal{U}^{(p)}$ and we set

$$m_p = \#\mathcal{U}^{(p)} \tag{2.5}$$

and

$$\mathcal{U} = \cup_{p=0}^{n} \mathcal{U}^{(p)} \,. \tag{2.6}$$

The additional inequality (2.4), together with (2.3), will give a localization of the essential spectrum in the semi-classical limit.

2.2 Spectral properties of $\Delta_{f,h}$.

We consider the case when Ω is a connected compact oriented Riemannian manifold or $\Omega = \mathbb{R}^n$. Note that in the first case $\mathcal{C}_0^\infty(\Omega; E) = \mathcal{C}^\infty(\Omega; E)$.

Proposition 2.2. *Under Assumption 2.1, there exist $h_0 > 0$ and $c_0 > 0$ such that the following properties are satisfied for any $h \in (0, h_0]$.*
i) *The Witten Laplacians $\Delta_{f,h}$ as an unbounded operator on $L^2(\Omega; \Lambda T^*\Omega)$ is essentially self-adjoint on $\mathcal{C}_0^\infty(\Omega; \Lambda T^*\Omega)$.*
ii) *The essential spectrum $\sigma_{ess}(\Delta_{f,h}^{(p)})$ is contained in $[c_0, +\infty)$.*
iii) *The range of the spectral projection $1_{[0,h^{3/2})}(\Delta_{f,h}^{(p)})$ has the dimension m_p for all $h \in (0, h_0]$.*
iv) *For any Borel subset E_h of $[0, h^{3/2})$*

$$1_{E_h}(\Delta_{f,h}^{(p+1)}) d_{f,h}^{(p)} u = d_{f,h}^{(p)} 1_{E_h}(\Delta_{f,h}^{(p)}) u \tag{2.7}$$

holds for any $u \in L^2(\Omega; \Lambda^p T^*\Omega)$ such that $d_{f,h}^{(p)} u \in L^2(\Omega; \Lambda^{p+1} T^*\Omega)$.
v) *In the case $\Omega = \mathbb{R}^n$, we have*

$$\left(0 \in \text{ Ker } \Delta_{f,h}^{(0)}\right) \Leftrightarrow \left(e^{-f/h} \in L^2(\mathbb{R}^n)\right)$$

vi) *In the case $\Omega = \mathbb{R}^n$, we have*

$$\left(e^{-f/h} \in L^2(\mathbb{R}^n)\right) \Rightarrow \left(\lim_{|x|\to\infty} f(x) = +\infty\right) \,.$$

Proof. The statements i), ii) and iii) are known in the case of a compact manifold (see [CFKS], [HelSj3]). In this case, we have of course no essential spectrum.

Let us check these three properties in the case $\Omega = \mathbb{R}^n$.
i) The operator

$$\Delta_{f,h} = -h^2\Delta + \Psi(x) = d_{f,h}d^*_{f,h} + d^*_{f,h}d_{f,h}$$

is non-negative on $\mathcal{C}_0^\infty(\mathbb{R}^n;\Lambda T^*\mathbb{R}^n)$ while the matrix-valued function $\Psi(x)$ is $\mathcal{C}^\infty$. By Simader's result (see [Sima], [Hel]), $\Delta_{f,h}$ is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^n;\Lambda T^*\mathbb{R}^n)$.
ii) The localization of the essential spectrum is a consequence of (2.3) and (2.4) which imply the existence of $C>0$ and K such that, for all $u \in \Lambda^p\mathcal{C}_0^\infty(\complement K)$,

$$\langle u \mid \Delta^{(p)}_{f,h}u\rangle \geq \langle u \mid \Delta^{(p)}_{0,h}u\rangle + \frac{1}{C}||u||^2 - Ch||u||^2 .$$

When $h < h_0$, with $h_0 = \frac{1}{2C^2}$, we get

$$\langle u \mid \Delta^{(p)}_{f,h}u\rangle \geq \frac{1}{2C}||u||^2 , \ \forall u \in \Lambda^p C_0^\infty(\complement K) ,$$

and ii) by using Persson's Lemma.
iii) The previous inequality combined with a simple partition of unity argument shows that any normalized eigenvector ψ_h associated with an eigenvalue λ_h in $[0, h^{3/2})$ of $\Delta^{(p)}_{f,h}$ has to be localized in a neighborhood of K. Indeed take a C^∞-partition of unity (χ_1,χ_2) such that $\chi_1 \in \mathcal{C}_0^\infty(\Omega)$ such that $\chi_1 \equiv 1$ in a neighborhood of K, $\chi_1^2+\chi_2^2 = 1$, and write

$$\lambda_h||\psi_h||^2 = \langle \chi_1\psi_h \,|\, \Delta^{(p)}_{f,h}\chi_1\psi_h\rangle + \langle \chi_2\psi_h \,|\, \Delta^{(p)}_{f,h}\chi_2\psi_h\rangle - h^2\sum_j \|\nabla\chi_j\psi_h\|^2 .$$

With the previous inequality and $\|\psi_h\| = 1$, we obtain

$$\|\chi_2\psi_h\|^2 \leq 2C(\lambda_h + Mh^2) \leq C'\, h^{3/2} ,$$

with $M = \max_{x\in\Omega, i=1,2}|\nabla\chi_i(x)|^2$. This implies $\lim_{h\to 0}\|\chi_1\psi_h\| = 1$ while we also have

$$\langle \chi_1\psi_h \,|\, \Delta^{(p)}_{f,h}\chi_1\psi_h\rangle \leq \lambda_h + Mh^2 \leq C''h^{3/2} .$$

Like in [CFKS], the introduction of a partition of unity subordinated to a covering of K by balls with radius $h^{2/5}$, combined with the min-max principle, implies that $\chi_1\psi_h$ lies within a distance of order $O(h^{3/2-6/5}) = O(h^{3/10})$ from an m_p dimensional space. This m_p dimensional space is spanned by truncated eigenvectors associated with the harmonic approximations at each critical points with index p. Since $\lim_{h\to 0}\|\psi_h - \chi_1\psi_h\| = 0$, this leads to

$$\dim \mathrm{Ran}\ 1_{[0,h^{3/2})}(\Delta^{(p)}_{f,h}) = m_p\ . \tag{2.8}$$

iv) If $\psi_h \in L^2(\Omega;\Lambda^p T^*\Omega)$ is an eigenvector of $\Delta^{(p)}_{f,h}$ with eigenvalue λ_h in $[0,h^{3/2})$, then we have $d^{(p)}_{f,h}\,\psi_h \in L^2(\Omega;\Lambda^{p+1}T^*\Omega)$ and $d^{(p-1),*}_{f,h}\,\psi_h \in L^2(\Omega;\Lambda^{p-1}T^*\Omega)$. Moreover according to (2.2), $d^{(p)}_{f,h}\,\psi_h$ satisfies

$$\Delta^{(p+1)}_{f,h} d^{(p)}_{f,h}\,\psi_h = \lambda_h d^{(p)}_{f,h}\,\psi_h \quad \text{in } \mathcal{D}'(\Omega;\Lambda^p T^*\Omega).$$

Since $\Delta^{(p+1)}_{f,h}$ is essentially self-adjoint on $\mathcal{C}_0^\infty(\Omega;\Lambda^{p+1}T^*\Omega)$, $d^{(p)}_{f,h}\psi_h$ belongs to the domain of $\Delta^{(p+1)}_{f,h}$. There are two possibilities : either $d^{(p)}_{f,h}\psi_h$ equals 0 or $d^{(p)}_{f,h}\psi_h$ is an eigenvector of $\Delta^{(p+1)}_{f,h}$ with eigenvalue λ_h. In any case we have $d^{(p)}_{f,h}\psi_h = 1_{\{\lambda_h\}}(\Delta^{(p+1)}_{f,h})d^{(p)}_{f,h}\psi_h$. The same can be done with $d^{(p-1),*}_{f,h}\psi_h$.
Let E_h be a Borel subset of $[0,h^{3/2})$. We set $F^{(p)}_{E_h} = \mathrm{Ran}\ 1_{E_h}(\Delta^{(p)}_{f,h})$. If v belongs to $F^{(p)}_{E_h}$, we write

$$v = \sum_{k=1}^N \alpha_k \psi_{k,h}, \quad \text{with } \Delta^{(p)}_{f,h}\psi_{k,h} = \lambda_{k,h}\psi_{k,h}, \quad \lambda_{k,h} \in E_h \subset [0,h^{3/2}).$$

We get

$$d^{(p)}_{f,h}v = \sum_{k=1}^N \alpha_k d^{(p)}_{f,h}\psi_{k,h}\ \in F^{(p+1)}_{E_h}.$$

If $v \in F^{(p)\perp}_{E_h}$ and $d^{(p)}_{f,h}v \in L^2(\Omega;\Lambda^{p+1}T^*\Omega)$, we have for all $\theta \in F^{(p+1)}_{E_h}$

$$\langle \theta \mid d^{(p)}_{f,h}v\rangle = \langle d^{(p),*}_{f,h}\theta \mid v\rangle = 0$$

because $d^{(p),*}_{f,h}\theta$ belongs to $F^{(p)}_{E_h}$ with the same argument. Hence $d^{(p)}_{f,h}v$ belongs to $F^{(p+1)\perp}_{E_h}$.
v) The equivalence is a consequence of the essential self-adjointness of $\Delta^{(0)}_{f,h}$

on $\mathcal{C}_0^\infty(\mathbb{R}^n)$ and of the fact that, when Ω is connected, the only distribution solutions of $d^{(0)}_{f,h}u = 0$ are the functions $c \exp -\frac{f}{h}$ $(c \in \mathbb{R})$.
vi) If $e^{-f/h}$ belongs to $L^2(\mathbb{R}^n)$, the Agmon estimates lead to

$$e^{-f(x)/h} \leq C_0 e^{-c_0|x|/h} \,,$$

which gives[2] the existence of $c_1 > 0$ such that

$$f(x) \geq c_1|x|, \text{ for } |x| \geq \frac{1}{c_1} \,. \tag{2.9}$$

$\square$

3 Strict saddle points and main assumption

One part of the analysis relies on a good labelling of local minima. This follows essentially the approach of Bovier-Gayrard-Klein in [BGKl], which is based on the notion of strict saddle point defined below. The labelling of the local minima was proposed by these authors and is one of the key points of their probabilistic approach. Their intuition was based on the notion of exit times for the stochastic dynamics and their idea was to enumerate the local minima according to the decreasing order of exit times.

3.1 Strict saddle points.

We consider the case when Ω is a compact connected oriented manifold or $\Omega = \mathbb{R}^n$.
When $\Omega = \mathbb{R}^n$, $\overline{\Omega}$ denotes the one-point-compactification $\Omega \sqcup \{\infty\}$.
For a closed set $F \subset \Omega$, $\overline{F}$ will denote its closure in $\overline{\Omega}$. For the sake of coherence, we keep Assumption 2.1 for the function f although some definitions could be extended to a more general case.

Definition 3.1.
a) *For any $E \subset \overline{\Omega}$, the set of connected components of E is denoted by* $\mathrm{Conn}(E)$.

[2]Note that we only use the existence, for fixed $h > 0$, of a gap. This gives actually a necessary condition for f for having a Poincaré inequality.

We remind that the connected components are non empty closed subsets relative to the induced topology on E and therefore compact if E is a closed subset of $\overline{\Omega}$.
b) *For any $A, B \subset \overline{\Omega}$, $H(A,B)$ denotes the quantity*

$$\begin{aligned} H(A,B) \; &= \inf \Big\{ c \in]-\infty, +\infty],\ \exists C \in \text{ Conn}\left(\overline{f^{-1}(]-\infty, c])}\right), \\ &\qquad\qquad\qquad\qquad C \cap A \neq \emptyset \textit{ and } C \cap B \neq \emptyset \Big\}\,. \end{aligned} \tag{3.1}$$

We first start with a simple result about $H(A,B)$.

Proposition 3.2. *When A and B are closed nonempty subsets of $\overline{\Omega}$, $H(A,B)$ is a minimum :*

$$\exists C \in \text{ Conn}\left(\overline{f^{-1}(]-\infty, H(A,B)])}\right),\ C \cap A \neq \emptyset \textit{ and } C \cap B \neq \emptyset.$$

Proof. It is done in several steps :
a) *For any $c \in \mathbb{R} \cup \{+\infty\}$ the number of connected component of $\overline{f^{-1}(]-\infty, c])}$ is finite.* More precisely it satisfies

$$\#\overline{f^{-1}(]-\infty, c])} \leq 1 + \#\mathcal{U}\ , \tag{3.2}$$

where $\mathcal{U}$ is the set of critical points of f. This implies in particular a uniform bound of this number.

First there is the possible connected component of $\{\infty\}$. The limiting case $c = +\infty$ gives $\overline{f^{-1}(]-\infty, +\infty])} = \overline{\Omega}$ which is connected.
It suffices to consider the case $c < +\infty$. Let us consider C in Conn $\left(\overline{f^{-1}(]-\infty, c])}\right)$ such that $\infty \notin C$. It is a closed subset of $\overline{\Omega}$ which does not contain ∞ and therefore a compact connected subset of Ω. If every point $x \in C$ is critical, then C is reduced to a single point which belongs to $\mathcal{U}$. If there exists $x_0 \in C$ such that $\nabla f(x_0) \neq 0$, then C contains a bounded connected component in Ω of $f^{-1}(]-\infty, c[)$, denoted by C_0. In this last case $\overline{C_0} \subset C$ is a compact subset of Ω such that $f\Big|_{\partial C_0} = c$. Then C_0 and therefore C contains a local minimum of f. So we have shown that any connected component of $\overline{f^{-1}(]-\infty, c])}$ which does not contain ∞ contains a critical point of f.
b) *For $c > c' > H(A,B)$, for any C' in* Conn $\left(\overline{f^{-1}(]-\infty, c'])}\right)$ *there exists C*

in $\mathrm{Conn}\left(\overline{f^{-1}(]-\infty,c])}\right)$ *such that* $C'\subset C$.
We first observe that $\overline{f^{-1}(]-\infty,c'])}\subset\overline{f^{-1}(]-\infty,c])}$ are not empty.
Now take $x_0\in C'$ and observe that the connected component of $\overline{f^{-1}(-\infty,c])}$ containing x_0 contains C'.

c) *For any decreasing sequence* $(c_n)_{n\in\mathbb{N}}$ *such that* $\lim_{n\to\infty}c_n=H(A,B)$, *there exists a decreasing sequence of closed connected subsets* $K_n\supset K_{n+1}$ *in* $\overline{\Omega}$ *such that*

$$K_n\in\mathrm{Conn}\left(\overline{f^{-1}(]-\infty,c_n])}\right),\quad K_n\cap A\neq\emptyset,\quad K_n\cap B\neq\emptyset\,.$$

Since $\#\,\mathrm{Conn}\left(\overline{f^{-1}(-\infty,c_0])}\right)$ is finite, there exists $K_0\in\mathrm{Conn}\left(\overline{f^{-1}(-\infty,c_0])}\right)$ such that the set

$$\left\{k\in\mathbb{N},\exists C\in\mathrm{Conn}\left(\overline{f^{-1}(]-\infty,c_k])}\right),\ C\cap A\neq\emptyset,\ C\cap B\neq\emptyset,\ C\subset K_0\right\}$$

is infinite.
Assume that $K_n\in\mathrm{Conn}\left(\overline{f^{-1}(-\infty,c_n])}\right)$ satisfies the above condition with K_0 replaced by K_n. The set $\mathcal{K}_{n+1}=\left\{C\in\mathrm{Conn}\left(\overline{f^{-1}(]-\infty,c_{n+1}]}\right),\ C\subset K_n\right\}$ is finite. For any $C\in\mathrm{Conn}\left(\overline{f^{-1}(]-\infty,c_k]}\right)$, $k\geq n+1$, such that $C\subset K_n$ there exists $C'\in\mathcal{K}_{n+1}$ such that $C\subset C'$. Hence we can choose $K_{n+1}\in\mathcal{K}_{n+1}$ such that

$$\Big\{k\in\mathbb{N},k\geq n+1,\exists C\in\mathrm{Conn}\left(\overline{f^{-1}(]-\infty,c_k]}\right),\\ C\cap A\neq\emptyset,\ C\cap B\neq\emptyset,\ C\subset K_{n+1}\}$$

is infinite with $K_{n+1}\subset K_n$. It satisfies $K_{n+1}\cap A\neq\emptyset$ and $K_{n+1}\cap B\neq\emptyset$.

d) *End of the proof.*
The sequence $(K_n)_{n\in\mathbb{N}}$ is a decreasing sequence of non empty compact connected subsets of $\overline{\Omega}$. Hence the intersection $K=\cap_{n\in\mathbb{N}}K_n$ is a non empty connected subset of Ω. Similarly the sequences $(K_n\cap A)_{n\in\mathbb{N}}$ and $(K_n\cap B)_{n\in\mathbb{N}}$ are decreasing sequences of non empty compact subsets of $\overline{\Omega}$. Hence $K\cap A$ and $K\cap B$ are not empty. Finally $K\setminus\{\infty\}\subset f^{-1}(]-\infty,c_n])$ for any $n\in\mathbb{N}$ and we get

$$K\subset\overline{f^{-1}(]-\infty,H(A,B)])}.$$

$\square$

Definition 3.3. *Under assumption 2.1, let A and B be two closed subsets of $\overline{\Omega}$. We say that Z is a set of strict saddle points for (A,B) if it is not empty and satisfies the following three conditions :*

$$(ssp1) \qquad Z \subset \left(\mathcal{U}^{(1)} \cap f^{-1}(\{H(A,B)\})\right) \cup \{\infty\}\ ,$$

$$(ssp2) \qquad Z \cap A = \emptyset \quad and \quad Z \cap B = \emptyset\ ,$$

$$(ssp3) \qquad \left\{C \in \mathrm{Conn}\left(\overline{f^{-1}(]-\infty, H(A,B)])} \setminus Z\right),\ C\cap A \neq \emptyset, C \cap B \neq \emptyset\right\} = \emptyset\ .$$

The word "strict" refers to the condition (ssp2).

Examples 3.4. *Here are simple examples which show why it is convenient to introduce the point ∞.*
a) *If f is a $\mathcal{C}^\infty$ function such that $f(-1) < 0$, $f(+1) < 0$ and $f(0) = 0$. Only with this information, one can say that the pair $A = \{-1\}$, $B = \{+1\}$, admits a set of saddle points without discussing the behaviour of f at infinity or the number of critical points. Indeed f admits a maximum on $]-1,1[$, $f(x_0) \geq 0$ and $H(A,B) \in [\max\{f(-1), f(+1)\}, f(x_0)]$. We can take $Z = \{+\infty\}$ if $H(A,B) < f(x_0)$ or $Z = \{x_0, +\infty\}$ if $H(A,B) = f(x_0)$.*
This argument can be extended in arbitrary dimension. By setting $M = \max f(A \cup B)$ for two compact subsets A, B of Ω. If A, B do not intersect a common connected component of $f^{-1}((-\infty, M])$, then (A,B) admits a set of strict saddle points (adapt the proof of Proposition 3.5 below).

b) *Consider a function on f on $\mathbb{R}$ which has three local maxima a $x = 0, \pm 2$, with $f(0) = 3$, $f(-2) = +1$ and $f(+2) = +2$, two local minima at $x = \pm 1$, $f(\pm 1) = 0$, and equals $-x^2$ for $|x| \geq 5$. We take first $A = \{-1\}$ and $B = \{+1\}$. Then we have $H(A,B) = +2$ and one can take $Z(A,B) = \{+2\}$ or $Z(A,B) = \{+2, +\infty\}$. Indeed in our analysis the interesting saddle points are at $x = +2$ and $x = -2$. The simplest way to introduce these points without entering into questions about the geometry of f near infinity which can be complicated in dimension $n > 1$ is by considering in this case $Z(A,B) = \{+2, \infty\}$ and by* *working with other pairs of sets* $A_1 = \{+1\}$, $B_1 = \{+\infty\}$ *(or* $B_1 = \{-1, +\infty\}$*) for which* $Z(A_1, B_1) = \{+2\}$ *and* $A_2 = \{-1\}$, $B_2 = \{+\infty\}$ *(or* $B_2 = \{+1, \infty\}$*) for which* $Z(A_2, B_2) = \{-2\}$. *This situation occurs only in the case* $\Omega = \mathbb{R}^n$ *with* $e^{-f/h} \notin L^2(\mathbb{R}^n)$.

The previous definition (more precisely (ssp3)) says that, if Z is a set of strict saddle points for (A, B), then any connected component of the subset $\overline{f^{-1}(]-\infty, H(A,B)])}$ joining A and B meets Z. In particular any continuous path γ from $[0,1]$ into $\overline{\Omega}$ such that $f(\gamma(t)) \leq H(A,B)$ when $\gamma(t) \neq \infty$ and $\gamma(0) \in A$ and $\gamma(1) \in B$, meets Z. The proof is by contradiction. Suppose $\sup_{t\in]0,1[} f(\gamma(t)) \leq H(A,B)$. Then $\gamma(t) \in \overline{f^{-1}(]-\infty, H(A,B)])} \setminus Z$ and the connected component of $\overline{f^{-1}(]-\infty, H(A,B)])} \setminus Z$ containing γ has non empty intersection with A and B in contradiction with (ssp3). In order to compare this rather abstract definition with the more usual Morse theory, it is useful to recall a few remarks coming from the local analysis of a Morse function.

Local structure of the level sets of a Morse function

First we observe that, near a non critical point x_0 of f, one can find a ball B_{x_0} around x_0 and a set of local coordinates such that

$$A_f^{<}(x_0) := \{f(x) < f(x_0)\} \cap B_{x_0} = \{y_1 < 0\} \cap B_{x_0}\ .$$

Secondly, if x_0 is a critical point of index p, then there exists a ball B_{x_0} around x_0 and a set of local coordinates centered at x_0 such that

$$A_f^{<}(x_0) = \left\{ -\sum_{\ell=1}^{p} y_\ell^2 + \sum_{\ell=p+1}^{n} y_\ell^2 < 0 \right\} \cap B_{x_0},$$

and

$$\begin{aligned} A_f^{\leq}(x_0) &:= \{f(x) \leq f(x_0)\} \cap B_{x_0} \\ &= \left\{ -\sum_{\ell=1}^{p} y_\ell^2 + \sum_{\ell=p+1}^{n} y_\ell^2 \leq 0 \right\} \cap B_{x_0}, \end{aligned}$$

We now observe that

1. When $p = 0$ (local minimum), $A_f^{<}(x_0)$ is empty and $A_f^{\leq}(x_0)$ is reduced to x_0.

2. When $p = 1$, $A_f^{<}(x_0)$ has two connected components and x_0 belongs to the closure of each of the two components. This property is crucial in the discussion of (ssp3).

3. When $p \geq 2$, $A_f^{<}(x_0)$ is (arcwise) connected.

So we can now prove the

Proposition 3.5. *If A and B are disjoint non empty subsets of local minima of f, then the pair (A, B) admits a set of strict saddle points.*

Proof. First note that $H(A,B) < +\infty$. We have to prove that a set C, belonging to $\mathrm{Conn}\left(\overline{f^{-1}(]-\infty, H(A,B)])}\right)$ and satisfying $C \cap A \neq \emptyset$, $C \cap B \neq \emptyset$ and $\infty \notin C$, contains a critical point z of index 1 in $f^{-1}(H(A,B))$ (i.e. $z \in \mathcal{U}^{(1)}$ and $f(z) = H(A,B)$). After this, we just take for Z the collection of such critical points by adding the point ∞ for possible connected component C such that $\infty \in C$.
If $\infty \notin C$, then C is a compact connected component of $f^{-1}(]-\infty, H(A,B)])$ in Ω. Since f is a Morse function, there are two possibilities, resulting from the previous local analysis of f and of the connectedness of C : Either it is reduced to one point which is a local minimum of f, or it is the closure of a finite union of bounded connected components Ω_i of $f^{-1}(]-\infty, H(A,B)[)$.
The first case cannot occur indeed because $C \cap A \neq \emptyset$ and $C \cap B \neq \emptyset$ forbids C to be reduced to one point. Hence we are reduced to the case

$$C = \cup_{i=1}^{N} \overline{\Omega_i}\ ,$$

where $\Omega_1, \ldots \Omega_N$ are bounded connected components of $f^{-1}(]-\infty, H(A,B)[)$ (note that N is smaller than the number of local minima m_0).
Every $x \in A \cap C$ (resp. $x \in B \cap C$) belongs to some Ω_i. The Ω_i are labelled such that for all $i \in \{1, \ldots, M\}$, $A \cap \Omega_i \neq \emptyset$ and for all $i \in \{M+1, \ldots, N\}$, $A \cap \Omega_i = \emptyset$. We have

$$A \cap C \subset \cup_{i=1}^{M} \Omega_i \quad \text{and} \quad B \cap C \subset \cup_{i=M+1}^{N} \Omega_i .$$

Since C is connected, we have

$$C \cap \overline{(\bigcup_{i=1}^{M} \Omega_i)} \cap \overline{\left(\bigcup_{j=M+1}^{N} \Omega_j\right)} \neq \emptyset.$$

Therefore, there exists $i \leq M$ and $j \geq M+1$ such that $C \cap \overline{\Omega_i} \cap \overline{\Omega_j} \neq \emptyset$. Assume $x_0 \in C \cap \overline{\Omega_i} \cap \overline{\Omega_j}$ and note that $i \neq j$ implies $f(x_0) = H(A, B)$. Then we observe that, if x_0 was not a critical point, then the local analysis shows that $\Omega_i = \Omega_j$ and $i = j$, in contradiction with the assumption.
Similarly the analysis of the connectedness of the set $A_f^{<}(x_0)$ at critical points excludes all critical points except the case $p = 1$.
Therefore a point $x_0 \in C \cap \overline{\Omega_i} \cap \overline{\Omega_j}$ with $i \leq M$ and $j \geq M + 1$ is a critical point of index 1.

□

On the uniqueness of the set of strict saddle points

It is not possible to give a satisfactory definition of a unique set of strict saddle points even in the case of Proposition 3.5. When there is a set of strict saddle points, the maximal (with respect to the inclusion) set Z which satisfies the three conditions of Definition 3.3 is well defined but may contain many irrelevant points. This is not accurate enough and it is more natural for our purpose to consider the minimal sets Z. But even in the framework of Proposition 3.5 the minimal sets of strict saddle points with respect to the inclusion are not unique : Simply consider the case when a path going from one local minimum x_1, $A = \{x_1\}$ to a local minimum x_2, $B = \{x_2\}$, $x_1 \neq x_2$, has to meet two distinct critical points of index 1, y_1 and y_2 with $f(y_1) = f(y_2) = H(A, B)$; then one can take $Z = \{y_1\}$; $Z = \{y_2\}$ or $Z = \{y_1, y_2\}$ but their intersection is empty.
However it is possible to define the property that the pair (A, B) admits a unique strict saddle point.

Definition 3.6. *Let A, B be closed nonempty disjoint subsets of $\overline{\Omega}$. The point*

$z \in \mathcal{U}^{(1)} \cup \{\infty\}$ is said to be a unique strict saddle point for the pair (A,B) if

$$\Big(\bigcap_{C\in\mathcal{C}(A,B)} C\Big) \cap \complement_{\overline{\Omega}} A \cap \complement_{\overline{\Omega}} B \cap \left[\left(\mathcal{U}^{(1)} \cap f^{-1}(H(A,B))\right) \cup \{\infty\}\right] = \{z\}$$

where $\mathcal{C}(A,B)$ denotes the set of closed connected sets $C \subset \overline{f^{-1}(]-\infty, H(A,B)])}$ such that $C \cap A \neq \emptyset$ and $C \cap B \neq \emptyset$.

We conclude this paragraph with the following remark :

Remark 3.7. *In the case $\Omega = \mathbb{R}^n$, assume $A = \{x_0\}$ and $B = \{x_1, \ldots, x_N, \infty\}$ where $x_0, x_1 \ldots x_N$ are local minima of f. We set $B' = \{x_1, \ldots, x_N\}$. There are two cases.*
1) *$H(\{x_0\}, \{\infty\}) > H(\{x_0\}, B')$:*
Then $H(A,B) = H(\{x_0\}, B')$ and the problem is reduced to the analysis of (A, B'). By Proposition 3.5 (A,B) admits a set Z of strict saddle points. Moreover, the connected component of $\overline{f^{-1}(]-\infty, H(A,B)])} \setminus Z$ which contains x_0 is relatively compact in Ω (i.e. bounded). This case occurs in particular when $\lim_{|x|\to\infty} f(x) = +\infty$.
2) *$H(\{x_0\}, \{\infty\}) \leq H(\{x_0\}, B')$:*
Then saying that (A,B) admits a set Z of strict saddle points is an assumption on the behaviour of f in a neighborhood of ∞. In this case also, the connected component of $\overline{f^{-1}(]-\infty, H(A,B)])} \setminus Z$ which contains x_0 is relatively compact in Ω (i.e. bounded).
So we have shown that, if it is stated that (A,B) admits a unique strict saddle point z, the connected component of $\overline{f^{-1}(]-\infty, H(A,B)])} \setminus \{z\}$ which contains x_0 is relatively compact in Ω (i.e. bounded) in both cases.

3.2 Main assumption, notations and first consequences.

The next assumption is essentially the one introduced by Bovier-Gayrard-Klein in [BGKl]. It will imply that each exponentially small eigenvalue of $\Delta^{(0)}_{f,h}$ is simple, with a different asymptotic behavior. We introduce the set $\mathcal{C}_0$ defined by

a) $\mathcal{C}_0 = \emptyset$ if Ω is a compact connected oriented Riemannian manifold.

b) $\mathcal{C}_0 = \emptyset$ if $\Omega = \mathbb{R}^n$ with $e^{-f(x)/h} \in L^2(\mathbb{R}^n)$.

c) $\mathcal{C}_0 = \{\infty\}$ if $\Omega = \mathbb{R}^n$ and $e^{-f(x)/h} \notin L^2(\mathbb{R}^n)$.

Assumption 3.8.

The function f satisfies Assumption 2.1. Moreover there exists a labelling of the local minima $\mathcal{U}^{(0)} = \left\{U_1^{(0)}, \ldots, U_{m_0}^{(0)}\right\}$ such that, by setting

$$\mathcal{C}_k = \left\{U_k^{(0)}, \ldots, U_1^{(0)}\right\} \cup \mathcal{C}_0 \,,$$

we have :

i) *For $k \geq 2$, $U_k^{(0)}$ is the unique minimizer of*

$$H(U, \mathcal{C}_k \setminus \{U\}) - f(U), \quad U \in \mathcal{C}_k \setminus \mathcal{C}_0.$$

ii) *For any $k \in \{1, \ldots m_0\}$ ($k \geq 2$ in the case $\mathcal{C}_0 = \emptyset$) the pair $\left(\{U_k^{(0)}\}, \mathcal{C}_{k-1}\right)$ admits a unique strict saddle point z_k^*.*

By its definition, the point z_k^*, with $k \geq 2$ if $\mathcal{C}_0 = \emptyset$ and $k \geq 1$ if $\mathcal{C}_0 \neq \emptyset$, has to be a critical point of index 1.

Definition 3.9. *(The map j)*
If these critical points of index 1 are numbered $U_j^{(1)}$, $j = 1, \ldots, m_1$, we define the application $k \to j(k)$ on $\{1, \ldots, m_0\}$ if $\mathcal{C}_0 \neq \emptyset$ and $\{2, \ldots, m_0\}$ if $\mathcal{C}_0 = \emptyset$ by

$$U_{j(k)}^{(1)} = z_k^*. \tag{3.3}$$

In the case when $\mathcal{C}_0 = \emptyset$, we set $j(1) = 0$, with the convention that $U_0^{(1)} \notin \Omega$ and $f(U_0^{(1)}) = +\infty$.

The cases $\mathcal{C}_0 = \emptyset$ and $\mathcal{C}_0 \neq \emptyset$ will be distinguished by $j(1) = 0$ or $j(1) \neq 0$.

Definition 3.10. *Under Assumption 3.8, consider for $k \in \{1, \ldots, m_0\}$ the set E_k defined by:*

a) *For $j(k) \neq 0$, E_k is the connected component of $U_k^{(0)}$ in*

$$\overline{f^{-1}(]-\infty, f(U_{j(k)}^{(1)})])} \setminus \{U_{j(k)}^{(1)}\} .$$

b) $E_1 = \Omega$ *if* $j(1) = 0$.

Proposition 3.11. *Under Assumption 3.8 and with Definition 3.10, the following properties are satisfied :*
a) *The sequence* $\left(f(U_{j(k)}^{(1)}) - f(U_k^{(0)})\right)_{k\in\{1,\ldots,m_0\}}$ *is strictly decreasing (with the convention* $f(U_0^{(1)}) = +\infty$*).*
b) *For $j(k) \neq 0$, E_k is a relatively compact subset of Ω and* $\overline{E_k} = E_k \cup \left\{U_{j(k)}^{(1)}\right\}$.
In any case, E_k is included in $f^{-1}(]-\infty, f(U_{j(k)}^{(1)})])$.
c) *For any $(k,j) \in \{1,\ldots,m_0\} \times \{1,\ldots,m_1\}$, the relation $U_j^{(1)} \in E_k$ implies*

$$\textit{either} \quad (j = j(k') \textit{ for some} \quad k' > k) \quad \textit{or} \quad j \notin j(\{1,\ldots,m_0\}).$$

d) *For any $k \neq k' \in \{1,\ldots,m_0\}$, the relation $U_{k'}^{(0)} \in E_k$ implies*

$$\left(k' > k \quad \textit{and} \quad f(U_{k'}^{(0)}) > f(U_k^{(0)})\right)$$

e) *The application $j : \{1,\ldots,m_0\} \to \{0,1,\ldots,m_1\}$ is injective.*

Proof. **a)** The condition **i)** of Assumption 3.8 gives

$$\begin{aligned}
f(U_{j(k)}^{(1)}) - f(U_k^{(0)}) &= H(U_k^{(0)}, \mathcal{C}_k \setminus \{U_k^{(0)}\}) - f(U_k^{(0)}) \\
&< H(U_{k-1}^{(0)}, \mathcal{C}_k \setminus \{U_{k-1}^{(0)}\}) - f(U_{k-1}^{(0)}) \\
&\leq H(U_{k-1}^{(0)}, \mathcal{C}_{k-1} \setminus \{U_{k-1}^{(0)}\}) - f(U_{k-1}^{(0)}) \\
&\leq f(U_{j(k-1)}^{(1)}) - f(U_{k-1}^{(0)}) ,
\end{aligned}$$

where the last inequality is an equality if $j(k-1) \neq 0$.
b) It is a rewriting of Remark 3.7.
c) Assume $U_{j(k')} \in E_k$.
In the case $j(k) = 0$, then $E_k = E_1 = \Omega$ and $U_{j(k')} \in \Omega$ implies $k' > 1$.
Consider now the case $j(k) \neq 0$. Since $U_{j(k)}^{(1)} \notin E_k$, one has $k \neq k'$. Moreover

the inequality $f(U^{(1)}_{j(k')}) \leq f(U^{(1)}_{j(k)})$ implies that the connected component of $\overline{f^{-1}(]-\infty, f(U^{(1)}_{j(k')})])}$, which contains $U^{(1)}_{j(k')}$ is contained in $\overline{E_k}$. Hence E_k contains $U^{(0)}_k$ and $U^{(0)}_{k'}$. Finally E_k is modified into a closed connected set $\hat{E}_k$ lying in $\overline{f^{-1}(]-\infty, f(U^{(1)}_{j(k)})])} \setminus \left\{U^{(1)}_{j(k)}\right\}$ in the following way. Take Morse coordinates around $U^{(1)}_{j(k)}$ and consider, for $\rho > 0$ small enough, $E_{k,\rho} := E_k \cap \{|x| \leq \rho\}$ and its radial projection on $E^{red}_{k,\rho} := E_k \cap \{|x| = \rho\}$. Then $\hat{E}_{k,\rho} := (E_k \setminus E_{k,\rho}) \cup E^{red}_{k,\rho}$ is closed and can be considered as the image of E_k by a continuous application. Hence it is connected. We have found a closed connected set $\hat{E}_{k,\rho}$ lying in $E_k \subset \overline{f^{-1}(]-\infty, f(U^{(1)}_{j(k)})])}$, which contains $U^{(0)}_k$, $U^{(0)}_{k'}$, $k' \neq k$ and does not contain $U^{(1)}_{j(k)}$. Therefore one cannot have $k \leq k'$ because this would contradict the assumption that $U^{(1)}_{j(k)}$ is the unique saddle point between $U^{(0)}_k$ and $\mathcal{C}_{k-1}$ (Assumption 3.8-ii) and Definition 3.6). Indeed the existence of another saddle point is obtained by using Proposition 3.5 by slightly increasing the value of $f(U^{(1)}_{j(k)})$. Hence, the only possibility is $k' > k$.

d) Assume $U^{(0)}_{k'} \in E_k$ with $k \neq k'$. By the same argument as for c), one then takes a closed connected set $C_{k,k'} \subset E_k \subset \overline{f^{-1}(]-\infty, f(U^{(1)}_{j(k)})])}$ such that $U^{(0)}_k, U^{(0)}_{k'} \in C_{k,k'}$ and $U^{(1)}_{j(k)} \notin C_{k,k'}$. This implies $k' > k$.

Assume now by contradiction that

$$\left\{k' > k,\ U^{(0)}_{k'} \in E_k \text{ and } f(U^{(0)}_{k'}) \leq f(U^{(0)}_k)\right\} \neq \emptyset\,,$$

and let k_0 be its smallest element.

We deduce from the existence of C_{k,k_0} as a closed connected subset of $E_k \subset \overline{f^{-1}(]-\infty, f(U^{(1)}_{j(k)})])}$ containing $U^{(0)}_k$ and $U^{(0)}_{k_0}$, the inequality

$$f(U^{(1)}_{j(k_0)}) = H(U^{(0)}_{k_0}, \mathcal{C}_{k_0-1}) \leq f(U^{(1)}_{j(k)}).$$

Since the connected component C of $U^{(1)}_{j(k_0)}$ in $\overline{f^{-1}(]-\infty, f(U^{(1)}_{j(k_0)})])}$ contains $U^{(0)}_{k_0}$ and a point in $\mathcal{C}_{k_0-1}$, it is contained in $\overline{E_k}$ and $\overline{E_k}$ contains a point of $\mathcal{C}_{k_0-1}$. This point cannot belong to $\mathcal{C}_0$: In the case $j(k) = 0$, $\mathcal{C}_0 = \emptyset$ and in the case $j(k) \neq 0$ it is a consequence of b).

Hence there exists $k_1 < k_0$ such that $U^{(0)}_{k_1} \in C \subset E_k$. Finally, the condition **i)** of Assumption 3.8 for k_0 gives

$$\begin{aligned} f(U^{(1)}_{j(k_0)}) - f(U^{(0)}_{k_0}) &= H(U^{(0)}_{k_0}, \mathcal{C}_{k_0-1}) - f(U^{(0)}_{k_0}) \\ &< H(U^{(0)}_{k_1}, \mathcal{C}_{k_0} \setminus \{U^{(0)}_{k_1}\}) - f(U^{(0)}_{k_1}) \\ &\leq f(U^{(1)}_{j(k_0)}) - f(U^{(0)}_{k_1}) \ , \end{aligned}$$

For the last inequality we used the existence of a connected set C containing $U^{(0)}_{k_1}$ and the point $U^{(0)}_{k_0} \in \mathcal{C}_{k_0} \setminus \{U^{(0)}_{k_1}\}$ such that $f(C) \in]-\infty, f(U^{(1)}_{j(k)})$, with the definition of $H(U^{(0)}_{k_1}, \mathcal{C}_{k_0} \setminus \{U^{(0)}_{k_1}\})$.
Hence we obtain

$$f(U^{(0)}_{k_1}) < f(U^{(0)}_{k_0}) \leq f(U^{(0)}_{k}) \ ,$$

with $k_1 < k_0$ and $U^{(0)}_{k_1} \in E_k$ in contradiction with the definition of k_0. Hence we have proved

$$\forall k' > k, (U^{(0)}_{k'} \in E_k) \Rightarrow \left(f(U^{(0)}_{k'}) > f(U^{(0)}_{k}) \right) \ .$$

e) First of all the value 0 is attained at most once, that is for $k = 1$, when $\mathcal{C}_0 = \emptyset$. Assume $j(k) = j(k') \neq 0$. The point $U^{(1)}_{j(k)} = U^{(1)}_{j(k')} \in \mathcal{U}^{(1)}$ is the unique strict saddle point for $(U^{(0)}_{k}, \mathcal{C}_{k-1})$ and for $(U^{(0)}_{k'}, \mathcal{C}_{k'-1})$.
Then we have

$$\begin{aligned} &\text{either} \quad E_k = E_{k'} \ , \\ &\text{or} \quad \exists k_1 < k',\ U^{(0)}_{k_1} \in E_k \quad \text{and} \quad \exists k_2 < k,\ U^{(0)}_{k_2} \in E_{k'} \ . \end{aligned}$$

According to d), the first case implies

$$k \leq k' \quad \text{and} \quad k' \leq k \ ,$$

while the second case gives

$$k \leq k_1 < k' \quad \text{and} \quad k' \leq k_2 < k \ .$$

Hence only the first case is possible with $k' = k$.

$\square$

Remark 3.12. *In the case $j(1) = 0$, since we have by definition $E_1 = \Omega$, the property d) in Proposition 3.11 says that $U^{(0)}_1$ is a global minimum for f.*

3.3 A generic case.

We check here that Assumption 3.8 is generically [3] verified when $\mathcal{C}_0 = \emptyset$ (that is $j(1) = 0$), that is when Ω is a connected oriented compact Riemannian manifold or when $\Omega = \mathbb{R}^n$ and $e^{-f(x)/h} \in L^2(\mathbb{R}^n)$. Remind that in this last case $\lim_{|x|\to\infty} f(x) = +\infty$. A Morse function $f \in \mathcal{C}^\infty(\Omega)$, with $\lim_{|x|\to\infty} f(x) = +\infty$ if $\Omega = \mathbb{R}^n$, **generically** has $\#\mathcal{U}$ distinct singular values. Moreover one can also assume that generically :

Assumption 3.13. *All the quantities $f(U_j^{(1)}) - f(U_\alpha^{(0)})$, for $j \in \{1, \ldots, m_1\}$ and $\alpha \in \{1, \ldots, m_0\}$ are distinct.*

Proposition 3.14. *Assumption 3.13 implies Assumption 3.8.*

Proof. We start with $m_0 = \#\mathcal{U}^{(0)}$ unlabelled local minima :

$$\mathcal{U}^{(0)} = \{\mathcal{U}_\alpha^{(0)},\ \alpha \in A\}\ , \quad \text{with } \#A = m_0\ .$$

For any subset $A' \subset A$, $\#A' \geq 2$, and any $\alpha \in A'$, the pair $(\{U_\alpha^{(0)}\}, \{U_{\alpha'}^{(0)},\ \alpha' \in A', \alpha' \neq \alpha\})$ admits a set of strict saddle points according to Proposition 3.5. Since the set $f^{-1}(\{H(\alpha, A' \setminus \{\alpha\})\})$ is bounded and contains at most one element of $\mathcal{U}$, it has to be a critical point of index 1 and the pair $(\{U_\alpha^{(0)}\}, \{U_{\alpha'}^{(0)},\ \alpha' \in A', \alpha' \neq \alpha\})$ admits a unique strict saddle point $U_{\alpha,A'}^{(1)}$.

The labelling of the local minima and the verification of Assumption 3.8 can now be done by reverse induction from $k = m_0$ to $k = 2$.

Once $U_{m_0}^{(0)}, \ldots, U_{k+1}^{(0)}$, $k \geq 2$, are known, we set

$$\mathcal{C}_k = \left\{U_\alpha^{(0)},\ \alpha \in A\right\} \setminus \{U_{m_0}^{(0)}, \ldots, U_{k+1}^{(0)}\} = \mathcal{C}_{m_0} \setminus \{U_{m_0}^{(0)}, \ldots, U_{k+1}^{(0)}\}.$$

The point $U_k^{(0)}$ is then chosen as the point in $\mathcal{C}_k$ which minimizes the quantity

$$f(U_{\alpha,\mathcal{C}_k}^{(1)}) - f(U_\alpha^{(0)}), \quad \alpha \in \mathcal{C}_k.$$

It is uniquely defined according to Assumption 3.13.

[3] By assuming that we are considering functions with no critical points outside a given regular compact domain D of Ω, a generic function is a function such that $f\big|_D$ belongs to some fixed G_δ set of $\mathcal{C}^\infty(D)$.

4 Cut-off functions and quasimodes.

4.1 Labelling of local minima and cut-off functions.

Let us first recall some notations and definitions. The Riemannian metric is denoted by dx^2 and the corresponding geodesic distance between two points $x, y \in \Omega$ by $d_\Omega(x, y)$.

The Agmon metric associated with the Witten Laplacian $\Delta_{f,h}$ is the degenerate metric $|\nabla f(x)|^2 \, dx^2$ and the corresponding distance between two points $x, y \in \Omega$ by $d_{\mathrm{Ag}}(x, y)$.

For $x \in \Omega$ and $\varepsilon > 0$, $B(x, \varepsilon)$ denotes the open ball for the geodesic distance

$$B(x, \varepsilon) = \{y \in \Omega, \; d_\Omega(y, x) < \varepsilon\} .$$

Having in mind the Definition 3.10 of the set E_k, it is then easy to show

Lemma 4.1. *There exists $\varepsilon_1 > 0$ such that the following properties are verified :*

i) *For any critical point $U \in \mathcal{U}$, with index p, there exist Morse coordinates $x = (x_1, \ldots, x_n)$ such that*

$$\forall x \in B(U, 4\varepsilon_1), \quad f(x) - f(U) = -x_1^2 \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_n^2 .$$

ii) *We have the lower bound :* $\min\{d_\Omega(U, U'), \; U, U' \in \mathcal{U}, \; U \neq U'\} \geq 10\,\varepsilon_1$.

iii) *For any $U \in \mathcal{U}$ and any $k \in \{1, \ldots, m_0\}$*

$$(U \not\in \overline{E_k}) \Rightarrow (d_\Omega(U, \overline{E_k}) \geq 10\,\varepsilon_1).$$

If $\overset{\circ}{E}_k$ denotes the interior of E_k and ∂E_k its boundary, the open set Ω_k is then defined as

$$\Omega_k = \overset{\circ}{E}_k \cup \left(\bigcup_{U \in \mathcal{U} \cap \partial E_k, \; U \neq U^{(1)}_{j(k)}} B(U, 3\,\varepsilon_1) \right) . \tag{4.1}$$

Its closure $\overline{\Omega_k}$ equals Ω when $j(k) = 0$ and equals the compact arcwise connected set

$$\overline{\Omega_k} = E_k \cup \{U^{(1)}_{j(k)}\} \cup \left(\bigcup_{U \in \mathcal{U} \cap \partial E_k, \; U \neq U^{(1)}_{j(k)}} \overline{B(U, 3\,\varepsilon_1)} \right)$$

when $j(k) \neq 0$.
The cut-off function $\chi_{k,\varepsilon}$, $k \in \{1, \ldots, m_0\}$, will be supported in a neighborhood of $\overline{\Omega_k}$ with some specific behaviour near $U^{(1)}_{j(k)}$, when $j(k) \neq 0$.
For $\varepsilon > 0$ and $\delta > 0$, $0 < \delta < \varepsilon < \varepsilon_1$, we introduce the set $\tilde{\Omega}_k(\varepsilon, \delta)$ defined by

$$\tilde{\Omega}_k(\varepsilon, \delta) = \left\{x \in \Omega, d_\Omega \left(x, \overline{\Omega_k} \setminus B(U^{(1)}_{j(k)}, \varepsilon)\right) < \delta\right\} \cup B(U^{(1)}_{j(k)}, \varepsilon).$$

Then there exists $C > 0$ and $\varepsilon_0 \in (0, \varepsilon_1]$ such that, for any fixed $\varepsilon \in (0, \varepsilon_0]$, one can associate $\delta_\varepsilon \in (0, \varepsilon)$ and $C_\varepsilon > 0$ so that the estimates

$$\forall x \in \tilde{\Omega}_k(\varepsilon, \delta) \setminus \tilde{\Omega}_k(\varepsilon, \delta/2)\ , \quad f(U^{(1)}_{j(k)}) + \tfrac{\delta}{C_\varepsilon} \leq f(x) \leq f(U^{(1)}_{j(k)}) + C_\varepsilon\ , \tag{4.2}$$

$$\forall x \in B(U^{(1)}_{j(k)}, \varepsilon)\ , \qquad \left|f(x) - f(U^{(1)}_{j(k)})\right| \leq C\varepsilon\ , \tag{4.3}$$

hold for any $\delta \in (0, \delta_\varepsilon]$.
The cut-off $\chi_{k,\varepsilon}$ is now chosen such that

$$\operatorname{supp} \chi_{k,\varepsilon} \subset \tilde{\Omega}_k(\varepsilon, \delta_\varepsilon) \quad \text{and} \quad \chi_{k,\varepsilon}\Big|_{\tilde{\Omega}_k(\varepsilon,\delta_\varepsilon/2)\setminus B(U^{(1)}_{j(k)},\varepsilon)} \equiv 1.$$

In the case $j(k) = 0$, our definition simply says $\chi_{k,\varepsilon} \equiv 1$ on Ω.
Around $U^{(1)}_{j(k)}$, the cut-off function $\chi_{k,\varepsilon}$ is chosen[4] so that $U^{(1)}_{j(k)} \notin \operatorname{supp} \chi_{k,\varepsilon}$ and

$$\forall x \in B(U^{(1)}_{j(k)}, \varepsilon),\ \left(\chi_{k,\varepsilon}(x) \neq 0 \text{ and } f(x) < f(U^{(1)}_{j(k)})\right) \Rightarrow (x \in \mathring{E}_k \subset \Omega_k). \tag{4.4}$$

Before we summarize the properties of the cut-off functions $\chi_{k,\varepsilon}$, $k \in \{1, \ldots, m_0\}$, we invite the reader to look at the three pictures which illustrate the various possibilities of the local shape of $\tilde{\Omega}_k(\varepsilon, \delta)$ and of $\operatorname{supp} \nabla\chi_{k,\varepsilon}$ in a neighborhood of $x_0 \in \partial E_k$. Asymptotically, that is for ε_1 and ε going to 0, geodesic balls are equivalent to ellipsoids in Morse coordinates (We simplified the picture by drawing circles instead).

[4]For further calculations, we will be more specific in Subsection 4.2 about the shape of this cut-off around $U^{(1)}_{j(k)}$.

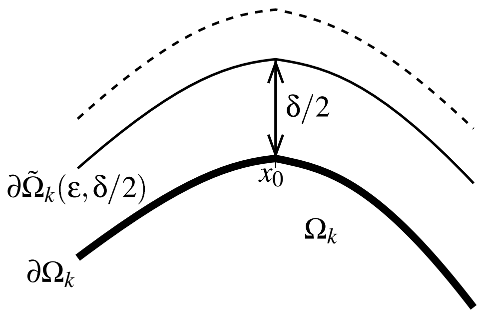

Figure 1: Case $x_0 \in \partial\Omega_k$, $\nabla f(x_0) \neq 0$. The support of $\nabla\chi_{k,\varepsilon}$ is localized around the dashed curve.

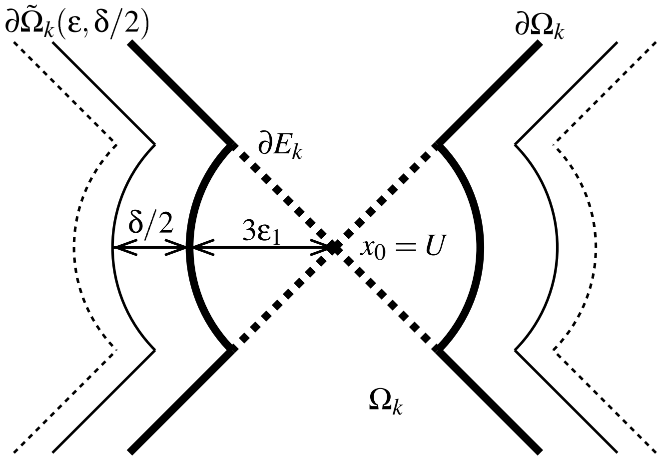

Figure 2: Case $x_0 = U \in \partial\Omega_k$, $\nabla f(U) = 0$ and $U \neq U^{(1)}_{j(k)}$. The support of $\nabla\chi_{k,\varepsilon}$ is localized around the dashed curve.

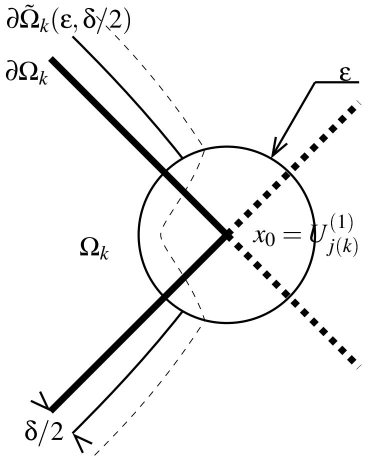

Figure 3: Case $x_0 = U^{(1)}_{j(k)}$. The support of $\nabla\chi_{k,\varepsilon}$ is localized around the dashed curve.

Proposition 4.2. *By taking $\delta = \delta_\varepsilon$ with $\varepsilon \in (0, \varepsilon_0]$, $0 < \varepsilon_0 \leq \varepsilon_1$ small enough, the cut-off functions $\chi_{k,\varepsilon}$, $k \in \{1, \dots, m_0\}$ satisfy the following properties :*

a) *If x belongs to $\operatorname{supp} \chi_{k,\varepsilon}$ and $f(x) < f(U^{(1)}_{j(k)})$, then $x \in \overset{\circ}{E}_k$.*

b) *There exist $C > 0$ and for any $\varepsilon \in (0, \varepsilon_0]$ a constant $C_\varepsilon > 0$ such that for $x \in \operatorname{supp} \nabla \chi_{k,\varepsilon}$:*

$$\begin{array}{rlll} \textit{either} & x \notin B(U^{(1)}_{j(k)}, \varepsilon) & \textit{and} & f(U^{(1)}_{j(k)}) + C_\varepsilon^{-1} \leq f(x) \leq f(U^{(1)}_{j(k)}) + C\varepsilon \\ \textit{or} & x \in B(U^{(1)}_{j(k)}, \varepsilon) & \textit{and} & \left| f(x) - f(U^{(1)}_{j(k)}) \right| \leq C\varepsilon. \end{array}$$

c) *For any $U \in \mathcal{U}$, $U \neq U^{(1)}_{j(k)}$, the distance $d_\Omega(U, \operatorname{supp} \nabla \chi_{k,\varepsilon})$ is bounded from below by $3\varepsilon_1 > 0$. If further $U \in \operatorname{supp} \chi_{k,\varepsilon}$, then $U \in E_k$.*

d) *If $U^{(0)}_{k'}$, for some $k' \in \{1, \dots, m_0\}$, belongs to $\operatorname{supp} \chi_{k,\varepsilon}$, then $k' \geq k$ and*

$$f(U^{(0)}_{k'}) > f(U^{(0)}_{k}), \quad f(U^{(1)}_{j(k')}) \leq f(U^{1}_{j(k)}), \quad \textit{if } k \neq k'.$$

e) *For any $j \in \{1, \dots, m_1\}$ such that $U^{(1)}_j \in \operatorname{supp} \chi_{k,\varepsilon}$:*

$$\begin{array}{rl} \textit{either} & j \notin j\left(\{1, \dots, m_0\}\right) \\ \textit{or} & j = j(k') \textit{ with } k' \geq k \textit{ and } U^{(0)}_{k'} \in \operatorname{supp} \chi_{k,\varepsilon}. \end{array}$$

Proof. **a)** is an immediate consequence of the local description of $\tilde{\Omega}_k(\varepsilon, \delta)$ in a neighborhood of $x_0 \in \partial E_k$.
b) is a consequence of the inequalities (4.2) and (4.3).
In **c)** the first statement is a consequence of the choice of ε_1 in Lemma 4.1. The second statement comes from the local description of $\tilde{\Omega}_k(\varepsilon, \delta)$ for $\delta > 0$ small enough.
d) is a consequence c) and Proposition 3.11-d).
e) is a consequence of c) and Proposition 3.11-c).

$\square$

4.2 Cut-off functions near saddle points.

We specify here the behaviour of the cut-off $\chi_{k,\varepsilon}$ in the ball $B(U^{(1)}_{j(k)},\varepsilon)$ with $j(k)\neq 0$ and $\varepsilon\in(0,\varepsilon_0]$, $\varepsilon_0>0$ small enough. We introduce like in ([HelSj3]-Section 2), the coordinates (y,z) which are adapted to the WKB-analysis of $\Delta^{(p)}_{f,h}$ near a critical point $U=U^{(p)}$ with index p (Actually we simply need the case $p=1$ here). We associate with this critical point U the stable (or incoming) manifold V_- and the unstable (or outgoing) manifold V_+ for ∇f, $\dim V_-=p$ and $\dim V_+=n-p$. We set

$$\Phi(x)=d_{\mathrm{Ag}}(x,U)\ ,$$

where d_{Ag} is the Agmon distance introduced in Subsection 4.1. In a neighborhood of $\mathcal{V}$ of U we have :

$$|f(x)-f(U)|\leq\Phi(x),\quad\forall x\in\mathcal{V},\tag{4.5}$$

and

$$(|f(x)-f(U)|=\Phi(x))\Leftrightarrow(x\in V_-\cup V_+)\ .\tag{4.6}$$

More precisely we have

$$\forall x\in V_\pm\cap\mathcal{V},\quad\Phi(x)=\pm\left(f(x)-f(U)\right)\ .$$

We now set for all $x\in\mathcal{V}$

$$g_+(x)=\Phi(x)-f(x)+f(U)\quad\text{and}\quad g_-(x)=\Phi(x)+f(x)-f(U).$$

The relation (due to the fact that Φ is locally a solution of the eikonal equation) in the neighborhood of U

$$|\nabla\Phi(x)|^2=|\nabla f(x)|^2\ ,\tag{4.7}$$

gives

$$\nabla g_+.\nabla g_-=0\ .$$

Moreover g_+ (resp. g_-) vanishes at order 2 on V_+ (resp. V_-) with a non degenerate transverse Hessian by taking $\mathcal{V}$ small enough. We also have

$$\begin{aligned}
&\nabla g_+ = 0 \quad \text{and} \quad \nabla g_- = 2\nabla f = 2\nabla\Phi \quad \text{on } V_+ \,,\\
&\nabla g_- = 0 \quad \text{and} \quad \nabla g_+ = -2\nabla f = 2\nabla\Phi \quad \text{on } V_- \,,
\end{aligned}$$

and ∇g_- (resp. ∇g_+) is tangent to V_+ (resp. V_-).
One first determines the coordinates $y_1, \ldots, y_p$ on V_- centered at U ($y_j(U) = 0$) such that the 1-forms $dy_1, \ldots dy_p$ define at U an orthonormal system of eigenvectors of $\operatorname{Hess} f(U)$ corresponding to its negative eigenvalues. Since g_- vanishes at order 2 on V_- with nondegenerate transverse Hessian **which has a fixed sign**, the coordinates y_j can be extended to a neighborhood of V_- as $\mathcal{C}^\infty$-solutions of

$$\nabla_{g_-} y_j = 0, \quad 1 \leq j \leq p \,.$$

Since ∇g_- is tangent to V_+, we have

$$y_j\Big|_{V_+} = 0, \quad 1 \leq j \leq p \,.$$

Moreover, any $\mathcal{C}^\infty$-function which solves $\nabla_{g_-} u = 0$ can be written as a function of $(y_1, \ldots, y_p)$. In particular we can write

$$g_+ = g_+(y_1, \ldots, y_p) \,.$$

Similarly the coordinates $z_{p+1}, \ldots, z_n$ are first defined on V_+ such that $z_j(U) = 0$ and $(dz_{p+1}(U), \ldots dz_n(U))$ is an orthonormal system of eigenvectors of $\operatorname{Hess} f(U)$ corresponding to positive eigenvalues. They are extended as solutions of

$$\nabla_{g_+} z_j = 0, \quad p+1 \leq j \leq n \,,$$

and satisfy : $z_j\Big|_{V_-} = 0, \quad p+1 \leq j \leq n$ and $g_- = g_-(z_{p+1}, \ldots, z_n)$. Since $g_\pm$ vanishes at order 2 and has a non degenerate transverse Hessian on $V_\pm$, the coordinates $(y_1, \ldots, y_p)$ and $(z_{p+1}, \ldots, z_n)$ can be replaced by Morse coordinates. If $\widehat{\lambda}_1(U) \leq \widehat{\lambda}_2(U) \ldots \leq \widehat{\lambda}_n(U)$ denote the eigenvalues of $\operatorname{Hess} f(U)$, we obtain coordinates (y, z) such that

$$\begin{aligned} f - f(U) &= \frac{1}{2}\left(-g_+(y_1, \ldots, y_p) + g_-(z_{p+1}, \ldots, z_n)\right) \\ &= \sum_{j=1}^{p} \frac{\widehat{\lambda}_j(U)}{2} y_j^2 + \sum_{j=p+1}^{n} \frac{\widehat{\lambda}_j(U)}{2} z_j^2 \,, \end{aligned}$$

and such that $(dy_1(U), \ldots, dz_n(U))$ is an orthonormal system of eigenvectors for Hess $f(U)$.

We will use such a set of coordinates in a neighborhood $\mathcal{V}$ of $U = U^{(1)}_{j(k)}$, $j(k) \neq 0$. Note that in this case $p = 1$, $V_- \cap \mathcal{V} = \{z_2 = \ldots = z_n = 0\} \cap \mathcal{V}$ and $V_+ \cap \mathcal{V} = \{y_1 = 0\} \cap \mathcal{V}$. The orientation of the y_1-axis V_- is chosen such that

$$\Omega_k \cap \mathcal{V} \subset \{y_1 < 0\} \cap \mathcal{V}.$$

The parameter $\varepsilon_0 > 0$ and for $\varepsilon \in (0, \varepsilon_0]$ the cut-off $\chi_{k,\varepsilon}$ are chosen such that :

i) The ball $B(U^{(1)}_{j(k)}, \varepsilon_0)$ is contained in $\mathcal{V}$.

ii) The support $\chi_{k,\varepsilon}$ does not meet V_+ :

$$\operatorname{supp} \chi_{k,\varepsilon} \cap B(U^{(1)}_{j(k)}, \varepsilon) \subset \{y_1 < 0\} \cap B(U^{(1)}_{j(k)}, \varepsilon).$$

iii) In a neighborhood

$$\mathcal{V}_- = \left\{ x \in B(U^{(1)}_{j(k)}, \varepsilon), \quad \max_{j=2,\ldots,n} |z_j(x)| \leq \nu_\varepsilon \right\}, \quad \nu_\varepsilon > 0 \,, \tag{4.8}$$

of $V_- \cap B(U^{(1)}_{j(k)}, \varepsilon)$, the function $\chi_{k,\varepsilon}$ only depends on y_1 :

$$\chi_{k,\varepsilon} = \chi_{k,\varepsilon}(y_1) \,.$$

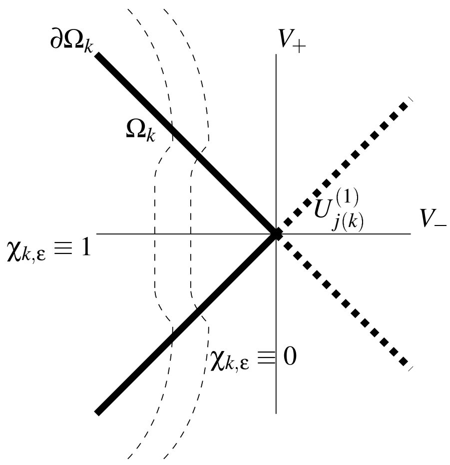

Figure 4: The support of $\nabla\chi_{k,\varepsilon}$ is localized between the dashed curves which coincide with $y_1 = \mathrm{Cst}$ near V_-.

4.3 Definition of quasimodes

The cut-off function $\chi_{k,\varepsilon}$ is used in the construction of quasi-modes for $\Delta_{f,h}^{(0)}$. The construction of quasi-modes for $\Delta_{f,h}^{(1)}$ will rely on the approximation by the Dirichlet problem in small balls around $U_j^{(1)}$, $j \in \{1, \ldots, m_1\}$. Let $\varepsilon_1 > 0$ be the positive radius independent of $\varepsilon > 0$ chosen in Definition 4.1. For each $j \in \{1, \ldots m_1\}$, we consider a normalized fundamental state u_j of the Witten Laplacian $\Delta_{f,h}^{(1)}$ in $B(U_j^{(1)}, 2\varepsilon_1)$ with Dirichlet boundary conditions on all components. The cut-off function $\theta_j \in \mathcal{C}_0^\infty(B(U_j^{(1)}, 2\varepsilon_1))$ is taken such that $\theta_j \equiv 1$ on $B(U_j^{(1)}, \varepsilon_1)$.

Note that the function $\chi_{k,\varepsilon}$ depends on $\varepsilon \in (0, \varepsilon_0]$, while θ_j is kept fixed like $\varepsilon_1 > 0$.

Definition 4.3.
For any $k \in \{1, \ldots, m_0\}$, the (ε, h)-dependent function $\psi_k^{(0)}$ is defined by

$$\psi_k^{(0)}(x) = \left\| \chi_{k,\varepsilon}(x) e^{-(f(x)-f(U_k^{(0)}))/h} \right\|^{-1} \chi_{k,\varepsilon}(x) e^{-(f(x)-f(U_k^{(0)}))/h} .$$

For any $j \in \{1, \ldots, m_1\}$, the h-dependent 1-form $\psi_j^{(1)}$ is defined by

$$\psi_j^{(1)}(x) = \left(\|\theta_j u_j\|^{-1}\right) \theta_j(x) u_j(x) .$$

For any $k \in \{1, \ldots, m_0\}$ we set

$$\lambda_k^{app}(\varepsilon, h) = \begin{cases} \left| \left\langle \psi_{j(k)}^{(1)} \mid d_{f,h}^{(0)} \psi_k^{(0)} \right\rangle \right|^2 & \text{if } j(k) \neq 0 \ , \\ 0 & \text{if } j(k) = 0 \ . \end{cases}$$

Remark 4.4. *For the sake of conciseness, we do not mention the (ε, h)- and h- dependence in the notations $\psi_k^{(0)}$ and $\psi_j^{(1)}$.*

5 Main result

Theorem 5.1.
Under Assumptions 2.1 and 3.8, there exist $\varepsilon_0 > 0$ and $\alpha > 0$, such that, for any $\varepsilon \in (0, \varepsilon_0]$,

$$\forall k \in \{1, \ldots, m_0\}, \quad \lambda_k(h) = \lambda_k^{app}(\varepsilon, h) \left(1 + \mathcal{O}_\varepsilon(e^{-\alpha/h})\right) \ .$$

Moreover, if $j(k) \neq 0$, there exists a sequence $(c_{k,m})_{m \in \mathbb{N}^*}$ independent of $\varepsilon \in (0, \varepsilon_0]$ such that

$$\lambda_k^{app}(\varepsilon, h) = \frac{h}{\pi} |\widehat{\lambda}_1(U_{j(k)}^{(1)})| \sqrt{\frac{\left|\det(\operatorname{Hess} f(U_k^{(0)}))\right|}{\left|\det(\operatorname{Hess} f(U_{j(k)}^{(1)}))\right|}} \\ \times \exp -\frac{2}{h} \left(f(U_{j(k)}^{(1)}) - f(U_k^{(0)}) \right) \times a_k(\varepsilon, h) \ ,$$

with

$$a_k(\varepsilon, h) \sim 1 + \sum_{k=1}^{\infty} c_{k,m} h^m \ .$$

6 Proof of Theorem 5.1

6.1 Quasimodal estimates.

In the next two sections, the parameter $\varepsilon_1 > 0$ is fixed, while ε_0 and $\varepsilon \in (0, \varepsilon_0]$ will be adapted in the different steps of the proof. **We shall denote by α a generic positive constant which is independent of $\varepsilon \in (0, \varepsilon_0]$.**

Proposition 6.1.
The system of (ε,h)-dependent functions $(\psi_k^{(0)})_{k\in\{1,\dots,m_0\}}$ of Definition 4.3 is almost orthogonal with

$$\left(\langle\psi_k^{(0)}\mid\psi_{k'}^{(0)}\rangle\right)_{k,k'\in\{1,\dots,m_0\}}=\mathrm{Id}_{\mathbb{C}^{m_0}}+\mathcal{O}_\varepsilon(e^{-\alpha/h})\,,$$

and there exists $\alpha>0$ and, for any $\varepsilon\in(0,\varepsilon_0]$, $C(\varepsilon)$ and $h_0(\varepsilon)$ such that, for any $h\in(0,h_0(\varepsilon)]$,

$$\langle\Delta_{f,h}^{(0)}\psi_k^{(0)}\mid\psi_k^{(0)}\rangle=\left\|d_{f,h}^{(0)}\psi_k^{(0)}\right\|^2\leq C(\varepsilon)e^{-2(f(U_{j(k)}^{(1)})-f(U_k^{(0)})-\alpha\varepsilon)/h}\,.$$

Proof. The almost orthogonality property is a direct consequence of Proposition 4.2-d) while the second estimate is given by

$$\langle\Delta_{f,h}^{(0)}\psi_k^{(0)}\mid\psi_k^{(0)}\rangle=\frac{\int_\Omega|\nabla\chi_{k,\varepsilon}(x)|^2\,e^{-2(f(x)-f(U_k^{(0)}))/h}\,dx}{\int_\Omega|\chi_{k,\varepsilon}(x)|^2\,e^{-2(f(x)-f(U_k^{(0)}))/h}\,dx}\,.$$

The denominator is seen of order $h^{n/2}$ by observing that $f(U_k^{(0)})$ is a non degenerate global minimum for $f\big|_{\operatorname{supp}\chi_{k,\varepsilon}}$ and using the Laplace integral method. The numerator is 0 in the case $j(k)=0$. In the case $j(k)\neq0$, the numerator is bounded by $C(\varepsilon)e^{-2\left(f(U_{j(k)}^{(1)})-f(U_k^{(0)})-C\varepsilon\right)/h}$ according to Proposition 4.2-b). This yields the result by taking $\alpha\leq C/2$.

$\square$

Corollary 6.2.
There exists $\varepsilon_0>0$ and $\alpha>0$ such that for any choice of ε in $(0,\varepsilon_0]$ the (ε,h)-dependent quasimodes $\psi_k^{(0)}$ satisfy the estimate

$$\langle\Delta_{f,h}^{(0)}\psi_k^{(0)}\mid\psi_k^{(0)}\rangle\leq C_\varepsilon e^{-\alpha/h}$$

for all $k\in\{1,\dots,m_0\}$.

Proposition 6.3.
The system of h-dependent 1-forms, $\left(\psi_j^{(1)}\right)_{j\in\{1,\dots,m_1\}}$ given in Definition 4.3 is orthonormal and there exists $\alpha>0$ independent of ε such that

$$\langle\Delta_{f,h}^{(1)}\psi_j^{(1)}\mid\psi_j^{(1)}\rangle=\mathcal{O}(e^{-\alpha/h})$$

for all $j\in\{1,\dots,m_1\}$.

Proof. The orthogonality is obvious with our choice of $\varepsilon_1 > 0$ in Lemma 4.1. The estimate is a consequence of Theorem 1.4 and Lemma 1.6 in [HelSj3] which says that the first eigenvalue of the Dirichlet Witten Laplacian $\Delta^{(1)}_{f,h}$ in $B(U_j^{(1)}, 2\varepsilon_1)$ is exponentially small and provides the Agmon type estimates for the first eigenvector

$$|u_j(x)| = \mathcal{O}_\eta\left(e^{-d_{\mathrm{Ag}}(x,U_j^{(1)})/h}\right)\cdot e^{\eta/h}\ ,\ \forall \eta > 0\ . \tag{6.1}$$

$\square$

Proposition 6.4. *There exist sequences $(c_{k,m})_{m\in\mathbb{N}^*}$, for $j(k)\neq 0$, such that the (ε,h)-dependent and h-dependent quasimodes $\psi_k^{(0)}$ and $\psi_j^{(1)}$ satisfy the identities*

$$\begin{aligned}
&\langle \psi_j^{(1)} \mid d^{(0)}_{f,h}\psi_k^{(0)}\rangle = 0 \quad \textit{if } j\neq j(k)\\
&\langle \psi_{j(k)}^{(1)} \mid d^{(0)}_{f,h}\psi_k^{(0)}\rangle \sim (-1)^{n-1}\frac{h^{1/2}}{\pi^{1/2}}|\widehat{\lambda}_1(U^{(1)}_{j(k)})|^{1/2}\left|\frac{\det(\mathrm{Hess}\, f(U_k^{(0)}))}{\det(\mathrm{Hess}\, f(U_{j(k)}^{(1)}))}\right|^{1/4}\\
&\qquad \times \exp -\frac{1}{h}\left(f(U^{(1)}_{j(k)}) - f(U^{(0)}_k)\right)\times\left[1+\sum_{m=1}^{\infty} c_{k,m}h^m\right]
\end{aligned}$$

for any $(k,j)\in\{1,\dots,m_0\}\times\{1,\dots,m_1\}$ as soon as $\varepsilon\in(0,\varepsilon_0]$.

Proof. The first statement is a consequence our choice of $\varepsilon_1 > 0$ and $\chi_{k,\varepsilon}$ which gives according to Proposition 4.2-c) $\mathrm{supp}\,\psi_j^{(1)}\cap\mathrm{supp}\,\nabla\chi_{k,\varepsilon} = \emptyset$. We conclude with $d^{(0)}_{f,h}\psi_k^{(0)} = C_{\varepsilon,h}\left(d^{(0)}\chi_{k,\varepsilon}\right)e^{-f/h}$.
We now need some accurate estimates for $\psi_k^{(0)}$ and $\psi^{(1)}_{j(k)}$ when $j(k)\neq 0$. Let us start with $\psi_k^{(0)}$.
We first need an expansion for the constant factor

$$\left\|\chi_{k,\varepsilon}e^{-\left(f(x)-f(U_k^{(0)})\right)/h}\right\|^2 = \int_\Omega |\chi_{k,\varepsilon}(x)|\, e^{-\frac{2(f(x)-f(U_k^{(0)}))}{h}}\, dx.$$

The Laplace method gives

$$\left\|\chi_{k,\varepsilon}e^{-\left(f(x)-f(U_k^{(0)})\right)/h}\right\|^2 \sim \frac{(\pi h)^{n/2}}{\left|\det \mathrm{Hess}\, f(U_k^{(0)})\right|^{1/2}}\left[1+\sum_{m=1}^{\infty}a_{k,m}h^m\right]$$

and we set

$$a_k(h) = \left\| \chi_{k,\varepsilon} e^{-\left(f(x)-f(U_k^{(0)})\right)/h} \right\|^{-1} = \frac{\left|\det \operatorname{Hess} f(U_k^{(0)})\right|^{1/4}}{\pi^{n/4}} [1+\mathcal{O}_\varepsilon(h)] \;,$$

with actually a complete expansion if necessary. Hence, the function $\psi_k^{(0)}$ and its differential $d_{f,h}^{(0)}\psi_k^{(0)}$ are equal to

$$\psi_k^{(0)}(x) = h^{-\frac{n}{4}}\, a_k(h) \chi_{k,\varepsilon}(x) e^{-\frac{(f(x)-f(U_k^{(0)}))}{h}} \tag{6.2}$$

and

$$d_{f,h}^{(0)}\psi_k^{(0)}(x) = h^{-\frac{n}{4}}\, a_k(h) (h d^{(0)}\chi_{k,\varepsilon})(x) e^{-\frac{(f(x)-f(U_k^{(0)}))}{h}} \;. \tag{6.3}$$

$$\begin{aligned}\left\langle \psi_{j(k)}^{(1)} \mid d_{f,h}^{(0)}\psi_k^{(0)} \right\rangle &= h^{1-\frac{n}{4}}\, a_k(h) \int_{B(U_{j(k)}^{(1)},\varepsilon)} (\psi_{j(k)}^{(1)} \mid d^{(0)}\chi_{k,\varepsilon})(x) e^{-\frac{(f(x)-f(U_k^{(0)}))}{h}}\, dx \\ &\quad + \mathcal{O}_\varepsilon\left(e^{-\frac{f(U_{j(k)}^{(1)})-f(U_k^{(0)})+\sigma_\varepsilon}{h}} \right), \quad \sigma_\varepsilon > 0.\end{aligned}$$

The three additional conditions i), ii) and iii) given in Subsection 4.2 for the cutoff function $\chi_{k,\varepsilon}$ combined with (4.6) permit to reduce the integration domain to the neighborhood $\mathcal{V}_-$, introduced in (4.8), of the stable manifold V_-. We obtain

$$\begin{aligned}\left\langle \psi_{j(k)}^{(1)} \mid d_{f,h}^{(0)}\psi_k^{(0)} \right\rangle &= h^{1-\frac{n}{4}}\, a_k(h) \int_{\mathcal{V}_-} (\psi_{j(k)}^{(1)} \mid {\chi'}_{k,\varepsilon} dy_1)(x) e^{-\frac{(f(x)-f(U_k^{(0)}))}{h}}\, dx \\ &\quad + \mathcal{O}_\varepsilon\left(e^{-\frac{f(U_{j(k)}^{(1)})-f(U_k^{(0)})+\sigma_\varepsilon}{h}} \right), \quad \sigma_\varepsilon > 0 \;.\end{aligned}$$

Theorem 2.5 in [HelSj3] says that in the coordinate system given in Subsection 4.2 there exists a WKB approximation

$$\omega \sim h^{-\frac{n}{4}} \exp -\frac{\Phi}{h} \;\; (\sum_{m=0}^{\infty} h^m \omega_m)$$

of $\psi_{(k)}^{(1)} = u_j$ in $B(U_{j(k)}^{(1)},\varepsilon)$ such that

$$\begin{aligned} &\left| e^{\Phi(x)/h}(u_j-\omega)(x)\right| = \mathcal{O}(h^\infty) \\ \text{and} \qquad &\omega_0 = (-1)^{n-1} \frac{\left|\det \operatorname{Hess} f(U_{j(k)}^{(1)})\right|^{1/4}}{\pi^{n/4}} \star (dz_2 \wedge \ldots \wedge dz_n) \quad \text{on } V_- \;.\end{aligned}$$

By setting $b_j(h) = (-1)^{n-1}\frac{\left|\det \mathrm{Hess}\, f(U^{(1)}_{j(k)})\right|^{1/4}}{\pi^{n/4}}$, we obtain

$$\left\langle \psi^{(1)}_{j(k)} \mid d^{(0)}_{f,h}\psi^{(0)}_k \right\rangle$$
$$= h^{1-\frac{n}{2}} a_k(h) b_j(h) \int_{\mathcal{V}_-} e^{-\frac{\Phi(x)+f(x)-f(U^{(0)}_k)}{h}} \left(\chi'_{k,\varepsilon}(y_1) + \mathcal{O}_\varepsilon(h)\right) dy_1 \wedge dz_2 \wedge \ldots \wedge dz_n \ .$$

and with $\Phi(x) + f(x) = g_-(z) + f(U^{(1)}_{j(k)})$

$$\left\langle \psi^{(1)}_{j(k)} \mid d^{(0)}_{f,h}\psi^{(0)}_k \right\rangle = h^{1-\frac{n}{2}}\, a_k(h) b_j(h) e^{-\frac{f(U^{(1)}_{j(k)})-f(U^{(0)}_k)}{h}} \times$$
$$\left[\int_{\mathcal{V}_-} e^{-g_-(z)/h}(\chi'_{k,\varepsilon}(y_1) + \mathcal{O}_\varepsilon(h)) dy_1 \wedge dz_2 \wedge \ldots \wedge dz_n\right] \ .$$

By Stokes formula the problem is reduced to the asymptotics of the integral

$$\int_{|z|\leq \nu} e^{-g_-(z)} dz_2 \wedge \ldots \wedge dz_n \quad \text{on } V_+ \ .$$

The final result

$$\left\langle \psi^{(1)}_{j(k)} \mid d^{(0)}_{f,h}\psi^{(0)}_k \right\rangle = h^{\frac{1}{2}}\, a_k(h) b_j(h) e^{-\frac{f(U^{(1)}_{j(k)})-f(U^{(0)}_k)}{h}} \times$$
$$\left[\frac{\pi^{\frac{n-1}{2}}}{\left|\widehat{\lambda}_2(U^{(1)}_{j(k)}) \ldots \widehat{\lambda}_n(U^{(1)}_{j(k)})\right|^{1/2}}\right] (1 + \mathcal{O}_\varepsilon(h)) \ .$$

is again an application of the Laplace method applied first to the main term and then to the remainder term. For the asymptotic expansion, one has to solve recursively the transport equations which determine the ω_m and apply the same trick with each term.

□

Corollary 6.5.
Let $\psi^{(0)}_k$ and $\psi^{(1)}_j$ denote the (ε, h)-dependent and h-dependent quasimodes of Definition 4.3. Assume that the 1-form $(w^{(1)}_j)_{j\in\{1,\ldots,m_1\}}$ satisfy

$$\left\| w^{(1)}_j - \psi^{(1)}_j \right\| = \mathcal{O}(e^{-\alpha/h}) \ ,$$

for some $\alpha > 0$ independent of $\varepsilon \in (0, \varepsilon_0]$. Then there exist $\varepsilon'_0 > 0$ and $\alpha' > 0$ such that, for all $\varepsilon \in (0, \varepsilon'_0]$, the estimates

$$\left|\langle w_j^{(1)} \mid d_{f,h}^{(0)}\psi_k^{(0)}\rangle\right| \leq C_\varepsilon e^{-\left(f(U_{j(k)}^{(1)})-f(U_k^{(0)})+\alpha'\right)/h} , \quad \text{if } j \neq j(k) , \tag{6.4}$$

and

$$\langle w_{j(k)}^{(1)} \mid d_{f,h}^{(0)}\psi_k^{(0)}\rangle = \langle \psi_{j(k)}^{(1)} \mid d_{f,h}^{(0)}\psi_k^{(0)}\rangle \left(1 + \mathcal{O}_\varepsilon(e^{-\alpha'/h})\right) , \tag{6.5}$$

hold for all $(k, j) \in \{1, \ldots, m_0\} \times \{1, \ldots, m_1\}$.

It is a straightforward consequence of Propositions 6.1 and 6.4 which give :

$$\left\|d_{f,h}^{(0)}\psi_k^{(0)}\right\| \leq C_\varepsilon e^{-\left(f(U_{j(k)}^{(1)})-f(U_k^{(0)})-\alpha''\varepsilon\right)/h} .$$

∎

6.2 Finite dimensional reduction

Our main tool here is the following consequence of the spectral theorem : For a non negative operator A and for $u \in D(A)$, we have

$$(\langle Au \mid u\rangle \leq a) \Rightarrow \left(\left\|1_{[b,+\infty)}(A)u\right\| \leq \frac{a}{b}\right) \tag{6.6}$$

for any $a, b > 0$.

This remark with Proposition 2.2 and the results of the previous Subsection 6.1 lead to the

Proposition 6.6.

There exist $\alpha, \alpha' > 0$ such that

$$1_{[0,h^{3/2})}(\Delta_{f,h}^{(\ell)}) = 1_{[0,e^{-\alpha/h})}(\Delta_{f,h}^{(\ell)})) \quad \text{for } \ell = 0, 1.$$

Moreover if one sets

$$\forall i \in \{1, \ldots, m_\ell\} , \quad v_i^{(\ell)} = 1_{[0,h^{3/2})}(\Delta_{f,h}^{(\ell)})\psi_i^{(\ell)}, \tag{6.7}$$

where the $\psi_i^{(\ell)}$ are the (ε, h)- and h- dependent quasimodes introduced in Definition 4.3, the system $\left(v_i^{(\ell)}\right)_{i\in\{1,\dots,m_\ell\}}$ is a basis of $F^{(\ell)}$ such that

$$\begin{aligned} &1) \quad \forall i \in \{1,\dots,m_\ell\}\,, \quad \left\|v_i^{(\ell)} - \psi_i^{(\ell)}\right\| \leq e^{-\alpha'/h} \\ &2) \quad V^{(\ell)} := \left(\langle v_i^{(\ell)}\, | v_{i'}^{(\ell)}\rangle\right)_{i,i'\in\{1,\dots,m_\ell\}} = \mathrm{Id}_{\mathbb{C}^{m_\ell}} + \mathcal{O}(e^{-\alpha'/h}). \end{aligned}$$

Remark 6.7. *Note that here again we forget the (ε, h)-dependence (resp. h-dependence) of the functions $v_k^{(0)}$ (resp. 1-forms $v_j^{(1)}$) in the notation. dependence*

Proof. Let $\ell \in \{0,1\}$ and $i \in \{1,\dots,m_\ell\}$. According to (6.6), Corollary 6.2 and Proposition 6.3, $\left\|1_{[h^{3/2}/2,+\infty)}(\Delta_{f,h}^{(\ell)})\psi_i^{(\ell)}\right\|$ is estimated by $e^{-\alpha'/h}$. The second estimate then comes from the almost orthonormality of $\left(\psi_i^{(\ell)}\right)_{i\in\{1,\dots,m_\ell\}}$. Since we know by Proposition 2.2-iii) that $F^{(\ell)}$ has dimension m_ℓ, the system $(v_i^{(\ell)})_{i\in\{1,\dots,m_\ell\}}$ is a basis of $F^{(\ell)}$. We conclude with

$$\langle \Delta_{f,h}^{(\ell)} v_i^{(\ell)} \mid v_i^{(\ell)}\rangle \leq \langle \Delta_{f,h}^{(\ell)} \psi_i^{(\ell)} \mid \psi_i^{(\ell)}\rangle \leq e^{-2\alpha/h}\ .$$

□

Definition 6.8. *The basis $(e_i^{(\ell)})_{i\in\{1,\dots,m_\ell\}}$ of $F^{(\ell)}$ is the orthonormal basis derived from $(v_i^{(\ell)})_{i\in\{1,\dots,m_\ell\}}$ by the Gram-Schmidt orthonormalization procedure*

$$e_i^{(\ell)} = \sum_{i'} \left[(V^{(\ell)})^{-1/2}\right]_{ii'} v_{i'}^{(\ell)}.$$

The $m_1 \times m_0$ matrix $\mathcal{M}$ is the matrix of[5] *$\beta_{f,h}^{(0)}$ in the bases $(e_k^{(0)})_{k\in\{1,\dots,m_0\}}$ and $(e_j^{(1)})_{j\in\{1,\dots,m_1\}}$. Its square $\mathcal{M}^*\mathcal{M}$ is called the interaction matrix.*

According to (2.7), the m_0 first eigenvalues of the Witten Laplacian $\Delta_{f,h}^{(0)} = d_{f,h}^{(0)*}d_{f,h}^{(0)}$ are the eigenvalues of the interaction matrix $\mathcal{M}^*\mathcal{M}$. Hence it is theoretically possible to determine the low lying eigenvalues of $\Delta_{f,h}^{(0)}$ by analyzing the matrix $\mathcal{M}$. The problem is that the coefficients of the matrix $\mathcal{M}$ are not known

[5] We recall from (1.2) that $\beta_{f,h}^{(0)}$ is defined from $F^{(0)}$ into $F^{(1)}$ by the restriction of $d_{f,h}^{(0)}$ to $F^{(0)}$.

at this level accurately enough in order to split the different exponentially small scales. One possibility would be to analyze the structure of resonant and weakly resonant wells in the spirit of [HelSj2]. Some indications are given in [HelNi]. We will see that here it is more convenient to work with the matrix

$$\mathcal{I} = \left(\langle v_j^{(1)} \mid \beta_{f,h}^{(0)} v_k^{(0)}\rangle\right)_{(j,k)\in\{1,\ldots,m_1\}\times\{1,\ldots,m_0\}}. \tag{6.8}$$

of the map $\beta_{f,h}^{(0)}$, written in the bases $(v_k^{(0)})_{k\in\{1,\ldots,m_0\}}$ in $F^{(0)}$ and $\left(v_j^{(1),*}\right)_{j\in\{1,\ldots,m_1\}}$ dual to $(v_j^{(1)})_{j\in\{1,\ldots,m_1\}}$ in $F^{(1)}$. This permits to use directly all the accurate information that we have on the quasimodes $\psi_i^{(\ell)}$. The fact that these bases are not orthonormal does not make any problem if one notices that the eigenvalues of $\mathcal{M}^*\mathcal{M}$ are indeed the squares of the singular values of $\beta_{f,h}^{(0)}$.

6.3 Singular values and induction.

The first eigenvalues $\lambda_k(h)$, $1\leq k\leq m_0$, of $\Delta_{f,h}^{(0)}$ are the squares of the singular values[6] $\mu_{m_0+1-k}(\mathcal{M})$ of $\mathcal{M}$. In other words,

$$\lambda_k(h) = \left[\mu_{m_0+1-k}\left(\beta_{f,h}^{(0)}\right)\right]^2 .$$

We will use the simple consequence of the Fan inequalities (see [Sim1], [GoKr]) :

Proposition 6.9. *For any matrices A and B such that,*

$$\max\left\{\|B\|, \left\|B^{-1}\right\|\right\} \leq 1+\rho ,$$

the singular values of A and AB satisfy

$$\frac{\mu_k(A)}{(1+\rho)} \leq \mu_k(AB) \leq (1+\rho)\mu_k(A)$$

and the same holds with AB replaced by BA.

Hence a little change of bases, induces a relative little change of the singular values and it is not necessary to work with orthonormal bases in order to

[6]The singular values $\mu_k(A)$ are numbered here as usual in the decreasing order with $\mu_1(A)=\|A\|$.

estimate the singular values.

For example, we have for any $k \in \{1, \dots, m_0\}$,

$$\mu_k(\beta^{(0)}_{f,h}) = \mu_k(\mathcal{M}) = \mu_k(\mathcal{I})\left(1 + \mathcal{O}(e^{-\alpha/h})\right)$$

where $\mathcal{I}$ is the matrix of the map $\beta^{(0)}_{f,h}$ introduced in (6.8).

We will construct by reverse induction on K, from m_0 down to $K = 1$ or $K = 0$, two bases $(v^{(0)}_{k,K})_{k\in\{1,\dots,m_0\}}$ of $F^{(0)}$ and of $F^{(1)}$ $(v^{(1)}_{j,K})_{j\in\{1,\dots,m_1\}}$ so that the next properties hold for $\varepsilon \in (0, \varepsilon_0]$ and some $\alpha > 0$ independent of ε :

1) The systems $(v^{(0)}_{k,K})_{K<k\leq m_0}$ and $(v^{(1)}_{j(k),K})_{K<k\leq m_0}$ are orthonormal.

We then set

$$F^{(0)}_K = \mathrm{Span}\left\{v^{(0)}_{k,K}, K < k \leq m_0\right\} \quad \text{and} \quad F^{(1)}_K = \mathrm{Span}\left\{v^{(1)}_{j(k),K}, K < k \leq m_0\right\} .$$

2) For $1 \leq k \leq K$, $v^{(0)}_{k,K}$ belongs to $\left(F^{(0)}_K\right)^{\perp}$ and for $j \notin \{j(k), K < k \leq m_0\}$, $v^{(1)}_{j,K}$ belongs to $\left(F^{(1)}_K\right)^{\perp}$.

3) The estimates

$$\forall i \in \{1, \dots, m_\ell\}, \quad \left\| v^{(\ell)}_{i,K} - \psi^{(\ell)}_i \right\| = \mathcal{O}_\varepsilon(e^{-\alpha/h})$$

hold for $\ell = 0, 1$.

4) For $K < k \leq m_0$, the equality

$$\beta^{(0)}_{f,h} v^{(0)}_{k,K} = \nu_k v^{(1)}_{j(k),K} \quad \text{and} \quad \Delta^{(0)}_{f,h} v^{(0)}_{k,K} = \nu_k^2 v^{(0)}_{k,K}$$

hold with

$$\nu_k = \langle \psi^{(1)}_{j(k)} \mid d^{(0)}_{f,h}\psi^{(0)}_k \rangle \left(1 + \mathcal{O}_\varepsilon(e^{-\alpha/h})\right) .$$

They imply, observing also that $\nu_k \neq 0$,

$$\Delta^{(\ell)}_{f,h} F^{(\ell)}_K \subset F^{(\ell)}_K, \qquad \ell \in \{0, 1\} .$$

5) For all $j \notin \{j(k), K < k \leq m_0\}$ and all $k \in \{1, \dots, K\}$, we have

$$\langle v^{(1)}_{j,K} \mid \beta^{(0)}_{f,h} v^{(0)}_{k,K} \rangle = \langle v^{(1)}_{j,K} \mid d^{(0)}_{f,h}\psi^{(0)}_k \rangle.$$

Remind that the $\psi_i^{(\ell)}$ and the $v_i^{(\ell)}$ depend on $h \in (0, h_0]$ and $\varepsilon \in (0, \varepsilon_0]$ while $\alpha > 0$ enters in the exponential estimates. The parameters $\varepsilon_0 > 0$ and $\alpha > 0$ belong to intervals which have to be reduced each time that one refers Corollary 6.5. This is done a finite number of times at each step of the induction.

Initialization: the case $K = m_0$.
We take $v_{k,m_0}^{(0)} = v_k^{(0)}$ and $v_{j,m_0}^{(1)} = v_j^{(1)}$ according to the definition of the previous section. The conditions 1), 2) and 4) are empty. The conditions 2) and 3) are given in Proposition 6.6. For the condition 5), we write

$$\begin{aligned}\langle v_j^{(1)} \mid \beta_{f,h}^{(0)} v_k^{(0)} \rangle &= \langle 1_{[0,h^{3/2})}(\Delta_{f,h}^{(1)}) v_j^{(1)} \mid d_{f,h}^{(0)} 1_{[0,h^{3/2})}(\Delta_{f,h}^{(0)}) \psi_k^{(0)} \rangle \\ &= \langle 1_{[0,h^{3/2})}(\Delta_{f,h}^{(1)}) v_j^{(1)} \mid d_{f,h}^{(0)} \psi_k^{(0)} \rangle = \langle v_j^{(1)} \mid d_{f,h}^{(0)} \psi_k^{(0)} \rangle .\end{aligned}$$

The recursion argument.
Assume that the result is true for $K > 0$ (or $K > 1$ if $j(1) = 0$). The conditions 1) and 4) say that the quantities $|\nu_k|$, $K < k \leq m_0$ are singular values of $\beta_{f,h}^{(0)}$ (ν_k^2 is an eigenvalue of $\Delta_{f,h}^{(0)}\big|_{F^{(0)}}$). Moreover the estimate

$$\nu_k = \langle \psi_{j(k)}^{(1)} \mid d_{f,h}^{(0)} \psi_k^{(0)} \rangle \left(1 + \mathcal{O}_\varepsilon(e^{-\alpha/h})\right) \tag{6.9}$$

and Proposition 6.4 imply

$$|\nu_k| \geq C_\varepsilon h^{1/2} e^{-(f(U_{j(K+1)}) - f(U_{K+1}^{(0)}))/h} \geq C_\varepsilon e^{-(f(U_{j(K)}) - f(U_K^{(0)}) - 2\alpha_1)/h} , \tag{6.10}$$

with α_1 independent of $\varepsilon > 0$.
Let us consider the dual basis $(v_{j,K}^{(1),*})$ in $F^{(1)}$. For $j = j(k)$, $K < k \leq m_0$, $v_{j,K}^{(1),*}$ equals $v_{j,K}^{(1)}$ and consequently

$$\left\| v_{j,K}^{(1),*} - \psi_j^{(1)} \right\| = \mathcal{O}_\varepsilon\left(e^{-\alpha/h}\right).$$

The matrix of $\beta_{f,h}^{(0)} : (F_K^{(0)})^\perp \to (F_K^{(1)})^\perp$ in the bases $(v_{k,K}^{(0)})_{1\leq k\leq K}$ and $(v_{j,K}^{(1),*})_{j\notin\{j(k), K<k\leq m_0\}}$ equals

$$\left(\langle v_{j,K}^{(1)} \mid \beta_{f,h}^{(0)} v_{k,K}^{(0)} \rangle \right)_{j\notin\{j(k),K<k\leq m_0\}, 1\leq k\leq K} . \tag{6.11}$$

The conditions 3) and 5) and Corollary 6.5 lead to

$$\left\|\beta_{f,h}\big|_{(F_K^{(0)})^\perp}\right\| = \mathcal{O}_\varepsilon(e^{-(f(U_{j(K)})-f(U_K^{(0)})-\alpha_1)/h}).$$

Hence the quantity $|\nu_k|$, $K<k\leq m_0$ are the first largest singular values of $\beta_{f,h}^{(0)}$,

$$\forall k\in\{K+1,\ldots,m_0\}\,,\quad |\nu_k| = \mu_{m_0+1-k}(\beta_{f,h}^{(0)}) = \sqrt{\lambda_k(h)},$$

and we have

$$\sqrt{\lambda_K(h)} = \mu_{m_0+1-K}(\beta_{f,h}^{(0)}) = \left\|\beta_{f,h}^{(0)}\big|_{(F_K^{(0)})^\perp}\right\|. \tag{6.12}$$

Let us now consider more carefully $\beta_{f,h}^{(0)}\big|_{(F_K^{(0)})^\perp}$ and its matrix (6.11) in the bases $(v_{k,K}^{(0)})_{1\leq k\leq K}$, $(v_{j,K}^{(1),*})_{j\notin\{j(k),K<k\leq m_0\}}$. With the same arguments as above relying on Corollary 6.5 and conditions 3) and 5), its coefficients have the form

$$\langle\psi_{j(K)}^{(1)}\mid d_{f,h}^{(0)}\psi_K^{(0)}\rangle\left(\delta_{j(K),j}\delta_{K,k}+\mathcal{O}_\varepsilon(e^{-\alpha_2/h})\right). \tag{6.13}$$

Since the two bases are $\mathcal{O}_\varepsilon(e^{-\alpha/h})$-close to orthonormal bases, we obtain

$$\sqrt{\lambda_K(h)} = \left|\langle\psi_{j(K)}^{(1)}\mid d_{f,h}^{(0)}\psi_K^{(0)}\rangle\right|(1+\mathcal{O}_\varepsilon(e^{-\alpha_3/h})).$$

We set

$$\nu_K = \frac{\langle\psi_{j(K)}^{(1)}\mid d_{f,h}^{(0)}\psi_K^{(0)}\rangle}{\left|\langle\psi_{j(K)}^{(1)}\mid d_{f,h}^{(0)}\psi_K^{(0)}\rangle\right|}\sqrt{\lambda_K(h)}\,. \tag{6.14}$$

We have

$$\beta_{f,h}^{(0)}v_{K,K}^{(0)} = \nu_K v_{j(K),K}^{(1),*}+\mathcal{O}_\varepsilon(\nu_K e^{-\alpha_4/h}). \tag{6.15}$$

We next define the new bases $(v_{k,K-1}^{(0)})$ and $(v_{j,K-1}^{(1)})$.
Of course we keep $v_{k,K-1}^{(0)} = v_{k,K}^{(0)}$ and $v_{j(k),K-1}^{(1)} = v_{j(k),K}^{(1)}$ for $K<k\leq m_0$.
We then take

$$\begin{aligned} v_{K,K-1}^{(0)} &= \left\|1_{\{\lambda_K\}}(\Delta_{f,h}^{(0)})v_{K,K}\right\|^{-1}1_{\{\lambda_K\}}(\Delta_{f,h}^{(0)})v_{K,K}\\ \text{and}\qquad v_{j(K),K-1}^{(1)} &= \frac{1}{\nu_K}\beta_{f,h}^{(0)}v_{K,K-1}^{(0)}.\end{aligned}$$

For $1\leq k\leq K-1$ and $j\notin\{j(k),K-1<k\leq m_0\}$, we take

$$\begin{aligned} v_{k,K-1}^{(0)} &= v_{k,K}^{(0)} - \langle v_{k,K}^{(0)}\mid v_{K,K-1}^{(0)}\rangle v_{K,K-1}^{(0)}\\ \text{and}\qquad v_{j,K-1}^{(1)} &= v_{j,K}^{(0)} - \langle v_{j,K}^{(1)}\mid v_{j(K),K-1}^{(1)}\rangle v_{j(K),K-1}^{(1)}.\end{aligned}$$

By construction the conditions 1), 2) and 4) are satisfied by these new bases. The condition 3) will be satisfied as well if $\left\|v_{K,K}^{(0)} - v_{K,K-1}^{(0)}\right\| = \mathcal{O}_\varepsilon(e^{-\alpha_5/h})$ holds. The identity (6.12) gives

$$\forall k \in \{1, \dots, K\}, \quad v_{k,K}^{(0)} = 1_{[0,\lambda_K]}(\Delta_{f,h}^{(0)}) v_{k,K}^{(0)}. \tag{6.16}$$

Moreover Corollary 6.5 yields

$$\forall k \in \{1, \dots, K-1\}, \forall j \in \{1, \dots, m_1\}, \quad \left|\langle v_{j,K}^{(1)} \mid \beta_{f,h}^{(0)} v_{k,K}^{(0)}\rangle\right| = \mathcal{O}_\varepsilon(\sqrt{\lambda_K} e^{-\alpha_6/h}).$$

Like in the proof of Proposition 6.6, we obtain for some $\alpha_7 > 0$

$$1_{[0,\lambda_K)}(\Delta_{f,h}^{(0)}) = 1_{[0,\lambda_K e^{-\alpha_7/h})}(\Delta_{f,h}^{(0)}). \tag{6.17}$$

We now write, by spectral decomposition and using (6.17) and (6.16),

$$\begin{aligned}\lambda_K \left\|1_{\{\lambda_K\}}(\Delta_{f,h}^{(0)}) v_{K,K}^{(0)}\right\|^2 + \mathcal{O}_\varepsilon(\lambda_K e^{-\alpha_7/h}) \left\|1_{[0,\lambda_K)}(\Delta_{f,h}^{(0)}) v_{K,K}^{(0)}\right\|^2 \\ = \langle \Delta_{f,h}^{(0)} v_{K,K}^{(0)} \mid v_{K,K}^{(0)}\rangle\end{aligned} \tag{6.18}$$

and observe that by (6.15)

$$\langle \Delta_{f,h} v_{K,K}^{(0)} \mid v_{K,K}^{(0)}\rangle = \left\|\beta_{f,h}^{(0)} v_{K,K}^{(0)}\right\|^2 = \lambda_K \left(1 + \mathcal{O}_\varepsilon(e^{-\alpha_4/2h})\right). \tag{6.19}$$

Hence we obtain

$$\left\|1_{\{\lambda_K\}}(\Delta_{f,h}^{(0)}) v_{K,K}^{(0)}\right\| = 1 + \mathcal{O}_\varepsilon(e^{-\alpha_8/h}).$$

We conclude with

$$\begin{aligned}\left\|1_{[0,\lambda_K)}(\Delta_{f,h}^{(0)}) v_{K,K}^{(0)}\right\|^2 = \left\|v_{K,K}^{(0)}\right\|^2 - \left\|1_{\{\lambda_K\}}(\Delta_{f,h}^{(0)}) v_{K,K}^{(0)}\right\|^2 \\ = \mathcal{O}_\varepsilon(e^{-2\alpha/h}) + \mathcal{O}_\varepsilon(e^{-2\alpha_8/h}).\end{aligned}$$

We have proved

$$\left\|v_{K,K}^{(0)} - v_{K,K-1}^{(0)}\right\| = \mathcal{O}_\varepsilon(e^{-\alpha_5/h}) .$$

This implies

$$\begin{aligned}\left\|\beta_{f,h}^{(0)} v_{K,K}^{(0)} - \nu_K v_{j(K),K-1}^{(1)}\right\| &= \left\|\beta_{f,h}^{(0)} v_{K,K}^{(0)} - \beta_{f,h}^{(0)} v_{K,K-1}^{(0)}\right\| \\ &= \left\|\beta_{f,h}^{(0)} 1_{[0,\lambda_K]}(\Delta_{f,h}^{(0)})(v_{K,K}^{(0)} - v_{K,K-1}^{(0)})\right\| \\ &= \mathcal{O}_\varepsilon(\sqrt{\lambda_K} e^{-\alpha_5/h}) ,\end{aligned}$$

while we have
$$\left\|\beta_{f,h}^{(0)}v_{K,K}^{(0)}-\nu_K v_{j(K),K}^{(1),*}\right\|=\mathcal{O}_\varepsilon(\nu_K e^{-\alpha_4/h}).$$
The almost orthonormality of $(v_{j,K}^{(1)})_{j\in\{1,\dots,m_0\}}$ inherited from the condition 3) and the almost orthogonality of $(\psi_j^{(1)})_{\{1,\dots,m_1\}}$ imply
$$\left\|v_{j(K),K}^{(1)}-v_{j(K),K}^{(1),*}\right\|=\mathcal{O}_\varepsilon(e^{-\alpha/2h}).$$
This yields
$$\left\|v_{j(K),K-1}^{(1)}-v_{j(K),K}^{(1)}\right\|=\mathcal{O}_\varepsilon(e^{-\alpha_9/h}).$$
Let us verify condition 5) for the new bases.

For $k\in\{1,\dots,K-1\}$ the construction of the new bases and the induction gives
$$\begin{aligned}v_{k,K-1}^{(0)}=v_{k,K}^{(0)}-\langle v_{k,K}^{(0)}\mid v_{K,K-1}^{(0)}\rangle v_{K,K-1}^{(0)} &= v_{k,m_0}^{(0)}-\sum_{K\le K'\le m_0}t_{k,K'}\,v_{K',K'-1}^{(0)}\\ &= v_k^{(0)}-\sum_{K\le K'\le m_0}t_{k,K'}\,v_{K',K-1}^{(0)}\,,\end{aligned}$$
with $t_{k,K'}:=\langle v_{k,K'}^{(0)}\mid v_{K',K'-1}^{(0)}\rangle$. Hence we get, with $v_k^{(0)}=1_{[0,h^{3/2})}(\Delta_{f,h}^{(0)})\,\psi_k^{(0)}$,
$$\begin{aligned}\beta_{f,h}^{(0)}\,v_{k,K-1}^{(0)} &= \beta_{f,h}^{(0)}\,v_k^{(0)}-\sum_{K\le K'\le m_0}t_{k,K'}\,\beta_{f,h}^{(0)}v_{K',K-1}^{(0)}\\ &= 1_{[0,h^{3/2})}(\Delta_{f,h}^{(1)})\,d_{f,h}^{(0)}\psi_k^{(0)}-\sum_{K\le K'\le m_0}t_{k,K'}\,\nu_{K'}\,v_{j(K'),K-1}^{(1)}.\end{aligned}$$
Meanwhile for $j\notin\{j(k),K-1<k\le m_0\}$, the vectors $v_{j,K-1}^{(1)}$ were constructed such that
$$v_{j,K-1}^{(1)}\in(F_{K-1}^{(1)})^\perp=\left(\mathrm{Span}\{v_{j(K),K-1}^{(1)},\dots,v_{j(m_0),K-1}^{(1)}\}\right)^\perp.$$
We obtain, for all $k\in\{1,\dots,K-1\}$ and all $j\notin\{j(k),K-1<k\le m_0\}$,
$$\begin{aligned}\langle v_{j,K-1}^{(1)}\mid\beta_{f,h}^{(0)}v_{k,K-1}^{(0)}\rangle &= \langle 1_{[0,h^{3/2})}\left(\Delta_{f,h}^{(1)}\right)v_{j,K-1}^{(1)}\mid d_{f,h}^{(0)}\psi_k^{(0)}\rangle\\ &= \langle v_{j,K-1}^{(1)}\mid d_{f,h}^{(0)}\psi_k^{(0)}\rangle.\end{aligned}$$

Stopping the induction :

When $j(1)\neq 0$, one continues the induction until the bases $(v_{k,0}^{(0)})$ and $(v_{j,0}^{(1)})$ are constructed. When $j(1) = 0$ ones stops the induction when the basis $(v_{k,1}^{(0)})$ and $(v_{j,1}^{(1)})$ are constructed. Indeed in this case we have $\beta_{f,h}^{(0)} v_1^{(0)} = 0$ and for all K, $1 \leq K \leq m_0$, $v_{1,K}^{(0)} = v_1^{(0)}$.

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