A NOTE ON SOLUTIONS TO A MODEL FOR LONG INTERNAL WAVES IN A ROTATING FLUID

Felipe Linares **Aniura Milanés**

In Memory of our friend Hebe Biagioni

Abstract

In this note we consider solutions of a nonlinear model for internal waves and its linearized version. We obtain a series of regularizing estimates for solutions of the linearized problem via the oscillatory integrals theory established in [9]. We also show a local smoothing effect for solutions of the nonlinear problem reminiscent of the one proved for solutions of the Benjamin-Ono equation in [15].

1 Introduction

In this note we will consider solutions to the initial value problem (IVP) for the equation

$$\partial_x(\partial_t u - \beta\mathcal{H}\partial_x^2 u + u\partial_x u) = \gamma u, \quad x, t \in \mathbb{R}, \tag{1.1}$$

where $\beta \cdot \gamma > 0$ and $\mathcal{H}$ denotes the Hilbert transform, i.e. $\mathcal{H}f = (-i\,\mathrm{sgn}(\xi)\widehat{f})^{\vee}$, and its linearized version.

When $\beta \neq 0$ and $\gamma > 0$ the equation above models the propagation of long internal waves in a deep rotating fluid (see [2], [3], [17]). In the context of shallow water the propagation of long waves in rotating fluid is described by the Ostrovsky equation (see [14], [12] and references therein).

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Before describing our results we first rewrite the equation (1.1) by using the antiderivative,

$$\partial_x^{-1} f(x) = \frac{1}{2}\left(\int_{-\infty}^{x} f(x')dx' - \int_{x}^{\infty} f(x')dx'\right) \tag{1.2}$$

(see [5]). Using this definition we then consider the following IVP

$$\begin{cases} \partial_t u - \beta\partial_x^2 \mathcal{H}u - \gamma\partial_x^{-1}u + u\partial_x u = 0, \quad x, t \in \mathbb{R}, \\ u(x,0) = u_0(x) \end{cases} \tag{1.3}$$

where u is a real valued function. We will also admit negative values for γ, so we simply assume $\beta\cdot\gamma \neq 0$. The equation in (1.3) can be seen as the well-known Benjamin-Ono (BO) equation,

$$\partial_t u - \partial_x^2 \mathcal{H}u + u\partial_x u = 0, \tag{1.4}$$

with an extra nonlocal term.

From the definition (1.2) it follows that $\partial_x^{-1} f = (\frac{\widehat{f}(\xi)}{i\xi})^{\vee}$. Then it is natural to define the function space X_s as

$$X_s = \{f \in H^s(\mathbb{R}) : \partial_x^{-1} f \in H^s(\mathbb{R})\}, \ s \in \mathbb{R}. \tag{1.5}$$

Our main purpose here is to obtain some regularizing effects for solutions of the linear problem associated to the equation in (1.3) and for solutions of the nonlinear problem (1.3).

For solutions of the linearized problem we establish Strichartz estimates, smoothing effects of Kato type and estimates of the maximal function type. The results will depend on the values of the parameters γ and β. We are able to prove global or local versions of the estimates previously mentioned according to the sign of the product $\beta \cdot \gamma$.

For several nonlinear dispersive equations the presence of these kind of smoothing effects have been useful to establish existence of solutions for non regular data. Thus it is an interesting problem to determine whether or not solutions to the linearized equations enjoy these properties.

Regarding the IVP (1.3), we have the following local well-posedness result.

Theorem 1.1. *Let $\beta \cdot \gamma \neq 0$, $s > 3/2$ and $u_0 \in X_s$. Then there exist $T = T(s, \|u_0\|_{X_s}) > 0$ and a unique solution u of the IVP* (1.3) *such that*

$$u \in C([0,T] : X_s(\mathbb{R})) \cup C^1([0,T] : X_{s-2}(\mathbb{R})).$$

Moreover, suppose that $u_0^n \to u_0 \in X_s(\mathbb{R})$ and u^n is the solution of (1.3) *with data $u^n(0) = u_0^n$. Then given $T' \in (0,T)$, there exists $N_0 = N_0(T')$ such that for $n \geq N_0$, u^n is defined in $[0,T']$ with*

$$\lim_{n\to\infty} \sup_{[0,T]} \|u^n(t) - u(t)\|_{X_s} = 0.$$

The proof of the first part of this theorem follows the same argument as the one given by Iorio [4] to establish local well-posedness for the IVP associated to the BO equation. The continuous dependence uses the Bona-Smith approximations [1]. Since these arguments are well known by now we do not give the proof of Theorem 1.1.

We will show that the solutions given by Theorem 1.1 are locally half derivative smoother than its initial data. This result was established by Ponce [15] for solutions of the BO equation. Our proof follows closely the arguments in [15].

The study of well-posedness for the IVP associated to the BO equation (1.4) has recently gained a lot of interest. In [18] Tao proved local and global well-posedness for data in $H^1(\mathbb{R})$ by using a gauge transformation. Kenig and Koenig [8] established local well-posedness for data in $H^s(\mathbb{R})$, $s > 9/8$ improving a previous work of Koch and Tzvetkov [11] (see also [4], [16]). The approach used by Kenig and Koenig seems to be more applicable to different situations than that for the BO equation. In a forthcoming paper we will use this framework to improve the local result stated in Theorem 1.1.

This note is organized as follows: the main results will be given in Section 2. The linear estimates will be proved in Section 3. In Section 4, the local smoothing effect for solutions of the IVP (1.3) will be established.

2 Main Results

In this section we will state the main results obtained in this work. To simplify our analysis, from now on we will consider $\beta = \pm 1$ and $\gamma = 1$. So we will denote $\phi_{\pm}(\xi) = \mp \xi|\xi| - \xi^{-1}$. Next we define the solution of the linear problem

$$\begin{cases} \partial_t v \mp \partial_x^2 \mathcal{H} v - \partial_x^{-1} v = 0, \quad x, t \in \mathbb{R}, \\ v(x,0) = v_0(x) \end{cases} \tag{2.6}$$

via the Fourier transform as

$$\mathcal{V}_{\pm}(t) v_0 = c \int_{\mathbb{R}} e^{ix\xi + it\phi_{\pm}(\xi)}\, \widehat{v}_0(\xi)\, d\xi. \tag{2.7}$$

The first set of estimates for solutions of the linear problem are the so-called Kato's smoothing effects.

Theorem 2.1. *Let $f \in L^2(\mathbb{R})$. Then*

$$\|D_x^{1/2} \mathcal{V}_+(t) f\|_{L_x^\infty L_T^2} \le c(1 + T^{1/2}) \|f\|_{L^2} \tag{2.8}$$

and

$$\|D_x^{1/2} \mathcal{V}_-(t) f\|_{L_x^\infty L_t^2} \le c\, \|f\|_{L^2}. \tag{2.9}$$

Another important regularizing effects satisfied by solutions of the linear problem (2.6) are given by the Strichartz estimates. In our case they are as follows.

Theorem 2.2. *Let $f \in L^2(\mathbb{R})$. Then*

$$\|\mathcal{V}_+(t) f\|_{L_t^q L_x^p} \le c\, \|f\|_{L^2}, \tag{2.10}$$

and

$$\|\mathcal{V}_-(t) f\|_{L_T^q L_x^p} \le c(1 + T^{1/q})\, \|f\|_{L^2}, \tag{2.11}$$

where $2/q = 1/2 - 1/p$, $p \ge 2$ (or $q = \frac{4}{\theta}$ and $p = \frac{2}{1-\theta}$, $\theta \in [0,1]$).

Remark 2.3. *In the above results we have either global or local estimates depending on the signs for $\phi_{\pm}$ and their first and second derivatives.*

To complete the set of estimates for solutions of the linearized problem we have the maximal function estimates.

Theorem 2.4. *Let $f \in H^{s_2}(\mathbb{R}) \cap \dot{H}^{-s_1}(\mathbb{R})$, $s_2 > 1/2$ and $s_1 > 1/4$. Then*

$$\|\mathcal{V}_{\pm}(t)f\|_{L_x^2 L_T^\infty} \leq c(1+T)^{1/2}(\|f\|_{\dot{H}^{-s_1}} + \|f\|_{H^{s_2}}). \tag{2.12}$$

Remark 2.5. *The Stricharz estimates and Kato's smoothing effects above are similar to those ones obtained for solutions of the linearized BO equation. The only obstruction given by the extra nonlocal term in equation* (1.3) *is the lack of global smoothing effects. This is reflected in the estimates* (2.8) *and* (2.11)*. On the other hand, due to that extra local term the maximal function estimates differ from the one proved for solutions of the linearized BO equation (see [10]).*

For solutions of the IVP (1.3) given by Theorem 1.1 we show the next local smoothing effect.

Theorem 2.6. *Let $s > 3/2$. If $u \in C([0,T] : X_s(\mathbb{R}))$ is the solution of the IVP* (1.3) *for $u_0 \in X_s$, then*

$$u \in L^2([0,T] : H_{loc}^{s+1/2}(\mathbb{R})). \tag{2.13}$$

Remark 2.7. *The proof of this result follows by using the same argument given by Ponce in [15]. For the sake of completeness we will give a sketch of it.*

3 Linear Estimates

We begin this section by proving the smoothing effect of Kato's type associated to solutions of the linear problem (2.6).

Proof of Theorem 2.1. To show (2.8) we consider $\psi \in C_0^\infty(\mathbb{R})$, $\psi \equiv 1$ for

$\xi \in [-3/4, 3/4]$ and $\operatorname{supp} \psi \subset [-1, 1]$. Then

$$
\begin{aligned}
D_x^{1/2} \mathcal{V}_+(t) f &= c \int_{\mathbb{R}} e^{ix\xi + it\phi_+(\xi)} \, |\xi|^{1/2} \, \widehat{f}(\xi) \, d\xi \\
&= c \int_{\mathbb{R}} e^{ix\xi + it\phi_+(\xi)} \, |\xi|^{1/2} \, \psi(\xi) \widehat{f}(\xi) \, d\xi \\
&+ c \int_{\mathbb{R}} e^{ix\xi + it\phi_+(\xi)} \, |\xi|^{1/2} \, (1 - \psi(\xi)) \widehat{f}(\xi) \, d\xi \\
&= A_1(x,t) + A_2(x,t).
\end{aligned}
\tag{3.14}
$$

Since $\phi'_+ \neq 0$ for $|\xi| > 1$ we can apply Theorem 4.1 in [9] to lead to

$$
\begin{aligned}
\sup_x \Big(\int_0^T |A_2(x,t)|^2 \, dt \Big)^{1/2} &\leq \sup_x \Big(\int_{\mathbb{R}} |A_2(x,t)|^2 \, dt \Big)^{1/2} \\
&\leq c \, \Big(\int \frac{|(1 - \psi(\xi)) \widehat{f}(\xi)|^2}{|\phi'_+(\xi)|} \, d\xi \Big)^{1/2} \leq c \, \|f\|_{L^2},
\end{aligned}
\tag{3.15}
$$

where in the last inequality we have used that $|1 - \psi(\xi)|^2 / |\phi'_+(\xi)| \in L^\infty$.

On the other hand, the Sobolev embedding gives

$$
\sup_x \, \Big(\int_0^T |A_1(x,t)|^2 \, dt \Big)^{1/2} \leq c \Big(\int_0^T \|D^{1/2}(\widehat{\psi} * f)\|^2_{H^1} \, dt \Big)^{1/2} \leq c \, T^{1/2} \|f\|_{L^2}.
\tag{3.16}
$$

Combining (3.14), (3.15) and (3.16) the estimate (2.8) follows.

To obtain inequality (2.9) we notice that ϕ_- belongs to a general class defined in [9]. Therefore from Theorem 4.1 in [9] the estimate (2.9) is deduced.

□

Next we prove the Strichartz estimates associated to solutions of the linear problem (2.6). The main tool will be the techniques used by Kenig, Ponce and Vega [9] to deal with oscillatory integrals.

Proof of Theorem 2.2. To establish estimate (2.10) we observe that

$$
|\phi''_+(\xi)| \geq 2.
$$

Thus

$$\|\mathcal{V}_+(t)f\|_{L^q_tL^p_x} \le \frac{1}{2}\Big(\int\limits_{\mathbb{R}} \|\int\limits_{\mathbb{R}} e^{it\phi_-(\xi)+ix\xi}|\phi''_+(\xi)|^{\theta/4}\widehat{f}(\xi)\,d\xi\|^q_{L^p}\,dt\Big)^{1/q}.$$

The inequality (2.10) follows by using Theorem 2.1 in [9] since ϕ_+ is in the general class defined in [9].

Next we prove (2.11). We consider $\psi \in C_0^\infty(\mathbb{R})$, $\psi \equiv 1$ for $\xi \in [-5/4, 5/4]$ and $\operatorname{supp}\psi \subset [-3/2, 3/2]$, then we have

$$\begin{aligned}\|\mathcal{V}_-(t)f\|_{L^q_TL^p_x} &\le c\Big(\int\limits_0^T \|\int\limits_{\mathbb{R}} e^{it\phi_-(\xi)+ix\xi}\widehat{f}(\xi)\psi(\xi)\,d\xi\|^q_{L^p}\,dt\Big)^{1/q}\\ &+ c\Big(\int\limits_0^T \|\int\limits_{\mathbb{R}} e^{it\phi_-(\xi)+ix\xi}\widehat{f}(\xi)(1-\psi(\xi))\,d\xi\|^q_{L^p}\,dt\Big)^{1/q}\\ &= B_1 + B_2.\end{aligned} \tag{3.17}$$

To estimate B_1 we use the Sobolev embedding, Holder's inequality and the regularity on ψ to obtain

$$B_1 \le c\Big(\int\limits_0^T \|f * \overset{\vee}{\psi}\|^q_{H^1}\,dt\Big)^{1/q} \le cT^{1/q}\|f\|_{L^2}. \tag{3.18}$$

Since $|\phi''_-(\xi)| \ne 0$ for $|\xi| > 3/2$ we rewrite B_2 and apply Theorem 2.1 in [9] to deduce that

$$B_2 = \Big(\int\limits_0^T \|\int\limits_{\mathbb{R}} e^{it\phi_-(\xi)+ix\xi}|\phi''_-(\xi)|^{\theta/4}\frac{\widehat{f}(\xi)(1-\psi(\xi))}{|\phi''_-(\xi)|^{\theta/4}}\,d\xi\|^q_{L^p}\,dt\Big)^{1/q} \le c\|f\|_{L^2} \tag{3.19}$$

where $\theta \in [0, 1]$.

Combining (3.17), (3.18) and (3.19) the inequality (2.11) follows.

□

To end this section we prove the maximal function estimates for solutions of the linear problem (2.6).

Proof of Theorem 2.4. Consider the following open covering of $\mathbb{R} - \{0\}$,

$$\Omega_k = (-2^{k+1}, -2^{k-1}) \cup (2^{k-1}, 2^{k+1}), k \in \mathbb{Z},$$

and a subordinated partition of unity $\{\varphi_k\}_{k=-\infty}^{\infty}$ and let

$$I_k^{\pm}(t,x) = c\int_{\mathbb{R}-\{0\}} e^{i(t\phi_\pm(\xi)+x\xi)}\varphi_k(\xi)d\xi.$$

As in [10], it suffices to prove that for any $k \in \mathbb{Z}$, there exists a function $H_k^{\pm} \in L^1(\mathbb{R})$ satisfying

$$|I_k^{\pm}(t,x)| \le H_k^{\pm}(x),$$

for any $x \in \mathbb{R}$ and $|t| \le T$ and such that

$$\|H_k^{\pm}\|_{L^1(\mathbb{R})} \le c(1+T)^a 2^{kb},$$

where a and b are some suitable constants.

Let us take $t \in [-T,T]$. We shall consider different cases.

3.1 Case 1: $k \ge 3$.

If $\xi \in \Omega_k$ then $|\phi_\pm'(\xi)| \le 6\cdot 2^k$. Then for $|x| > 12\cdot 2^k T$, we have $|t\phi_\pm'(\xi)+x\xi| > \frac{1}{2}|x| > \frac{1}{3}|x|$.

Assume that $12\cdot 2^k T > 1$ and let us consider a function $h \in C^\infty(\mathbb{R})$ such that supp $h \subset \left\{\xi : |t\phi_\pm'(\xi)+x| \le \frac{1}{2}|x|\right\}$ and that equals one in $\left\{\xi : |t\phi_\pm'(\xi)+x| \le \frac{1}{3}|x|\right\}$. Performing two integrations by parts and using the remarks above we obtain that when $|x| > 12\cdot 2^k T$,

$$\left|\int_{\mathbb{R}-\{0\}} e^{i(t\phi_\pm(\xi)+x\xi)}\varphi_k(\xi)(1-h(\xi))d\xi\right| \le c\frac{2^k}{|x|^2}.$$

If $\xi \in \Omega_k \cap \left\{\xi : |t\phi_\pm'(\xi)+x| \le \frac{1}{2}|x|\right\}$, we have that

$$\begin{aligned} |t\phi_\pm''(\xi)| &= 2\frac{|t|}{|\xi|}\left|\frac{\xi^3\pm 1}{\xi^2}\right| && (3.20)\\ &\ge \frac{|t\phi_\pm'(\xi)|}{4|\xi|} && (3.21)\\ &\ge \frac{1}{8}|\,x|\,2^{-k}. && (3.22)\end{aligned}$$

Where in the second line we have used that if $|\xi| \ge 3/2$ then

$$2(|\xi|^3-1) \ge \frac{1}{4}(2|\xi|^3+1). \tag{3.23}$$

Now we can use Van der Corput lemma to get,

$$\left|\int_{\mathbb{R}-\{0\}} e^{i(t\phi_\pm(\xi)+x\xi)}\varphi_k(\xi)h(\xi)d\xi\right| \le c\,\frac{2^{k/2}}{|x|^{1/2}}.$$

Thus for $12\cdot 2^k T > 1$, the theorem follows by choosing

$$H_k^\pm(x) = \begin{cases} 2^k & ,\ |x|\le 1 \\ \frac{2^k}{|x|^2} + \frac{2^{k/2}}{|x|^{1/2}} & ,\ 1<|x|\le 12\cdot 2^k T \\ \frac{2^k}{|x|^2} & ,\ |x| > 12\cdot 2^k T, \end{cases}$$

so that $\|H_k^\pm\|_{L^1(\mathbb{R})} \le c(1+T)^{1/2}2^k$. Otherwise we may set

$$H_k^\pm(x) = \begin{cases} 2^k & ,\ |x|\le 1 \\ \frac{2^k}{|x|^2} & ,\ |x|>1, \end{cases}$$

with $\|H_k^\pm\|_{L^1(\mathbb{R})} \le c2^k$.

3.2 Case 2: $k \le -2$.

Now $|\phi_\pm'(\xi)| \le 5\cdot 2^{-2k}$ if $\xi\in\Omega_k$.

To estimate I_k^-, we do not have inequalities such as (3.20)-(3.22), but we can use that for $\xi\in\Omega_k$, $|\xi|\le\frac{1}{4^{1/3}}$ and then

$$\begin{aligned} |\phi_-''(\xi)| &= \frac{2}{|\xi|^3}(1-|\xi|^3) \\ &\ge 2+\frac{1}{|\xi|^3} \\ &= \frac{|\phi_-'(\xi)|}{|\xi|}. \end{aligned}$$

Similarly as in the previous case we define

$$H_k^\pm(x) = \begin{cases} 2^k & ,\ |x|\le 1 \\ \frac{2^k}{|x|^2} + \frac{2^{k/2}}{|x|^{1/2}} & ,\ 1<|x|\le 10\cdot 2^{-2k} T \\ \frac{2^k}{|x|^2} & ,\ |x| > 10\cdot 2^{-2k} T, \end{cases}$$

for $10\cdot 2^{-2k}T>1$, with $\|H_k^\pm\|_{L^1(\mathbb{R})} \le c(1+T)^{1/2}2^{-k/2}$ and

$$H_k^\pm(x) = \begin{cases} 2^k & ,\ |x|\le 1 \\ \frac{2^k}{|x|^2} & ,\ |x|>1, \end{cases}$$

with $\|H_k^\pm\|_{L^1(\mathbb{R})} \le c2^k$ otherwise.

3.3 Case 3: $-1 \le k \le 2$.

In this case we have that $|\phi'_\pm(\xi)| \le 32$ and we can take

$$H_k^\pm(x) = \begin{cases} 1 & , |x| \le 64T \\ \frac{1}{|x|^2} & , |x| > 64T, \end{cases}$$

and so $\|H_k^\pm\|_{L^1(\mathbb{R})} \le c(1+T)$ for $64T > 1$, and

$$H_k^\pm(x) = \begin{cases} 1 & , |x| \le 1 \\ \frac{1}{|x|^2} & , |x| > 1, \end{cases}$$

with $\|H_k^\pm\|_{L^1(\mathbb{R})} \le c$ otherwise.

These estimates lead to the result. For the details, see [6].

□

4 Nonlinear Estimates

In this section we establish the local smoothing effect advertised in Theorem 2.6.

We need the following commutator estimates due to Kato and Ponce [7]

Lemma 4.1. *Let* $J^s = (1-\partial_x^2)^{s/2}$. *If* $s \ge 0$, $1 < p < \infty$, $f, g \in \mathcal{S}(\mathbb{R})$, *then there exists a constant* $c = c(s, n, p)$ *such that*

$$\|[J^s, f]g\|_p \le c\left\{\|\nabla f\|_{p_1}\|J^{s-1}g\|_{p_2} + \|J^s f\|_{p_3}\|g\|_{p_4}\right\} \tag{4.24}$$

and

$$\|J^s(fg)\|_p \le c\{\|f\|_{p_1}\|J^s g\|_{p_2} + \|J^s f\|_{p_3}\|g\|_{p_4}\} \tag{4.25}$$

where $1 < p_2,\ p_3 < \infty$ *and* $\dfrac{1}{p} = \dfrac{1}{p_1} + \dfrac{1}{p_2} = \dfrac{1}{p_3} + \dfrac{1}{p_4}$.

Here $[\cdot,\cdot]$ denotes the commutator $[A, B] = AB - BA$.

Applying the operator J^s to the equation in (1.3), multiplying it by $J^s u$, integrating and using Lemma 4.1 we can obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_s^2 \le c\|\partial_x u(t)\|_{L^\infty}\|u(t)\|_s^2.$$

On the other hand, using a similar argument we have that

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^{-1}u(t)\|_s^2 \leq c\|u(t)\|_{L^\infty}\|u(t)\|_s^2.$$

We notice that in these computations the operator ∂_x^{-2} appears so we need to use an analogous argument as the one employed in [13]. Hence we get the *a priori* estimate

$$\|u(t)\|_{X_s}^2 \leq c\|u_0\|_{X_s}^2 \exp\Big(\int_0^t \|u(t')\|_{1,\infty}\,dt'\Big) \tag{4.26}$$

which will be useful in what follows.

Proof of Theorem 2.6. Let $\phi:\mathbb{R}\mapsto\mathbb{R}$, increasing, $\phi\in C^\infty(\mathbb{R})$, such that $\phi'\in C_0^\infty(\mathbb{R})$. Let $u(t)$ be a solution of the equation

$$\partial_t u - \mathcal{H}\partial_x^2 u + (u\partial_x u) - \partial_x^{-1}u = 0. \tag{4.27}$$

It follows that

$$\partial_t\mathcal{H}u + \partial_x^2 u + \mathcal{H}(u\partial_x u) - \mathcal{H}\partial_x^{-1}u = 0. \tag{4.28}$$

Formally we obtain

$$\begin{aligned}\int\{\partial_t J^s u J^s u + \partial_t J^s\mathcal{H}u J^s\mathcal{H}u - J^s\mathcal{H}\partial_x^2 u J^s u + J^s\partial_x^2 u J^s\mathcal{H}u\\ + J^s(u\partial_x u)J^s u + J^s\mathcal{H}(u\partial_x u)J^s\mathcal{H}u\\ - J^s u J^s\partial_x^{-1}u - J^s u J^s\mathcal{H}\partial_x^{-1}u\}\phi(x)\,dx = 0.\end{aligned} \tag{4.29}$$

To make rigorous this identity and the ones below we have to make use of the continuous dependence of the solutions given in Theorem 1.1.

Now we proceed to estimate each term in (4.29). The first two terms in (4.29) can be written as

$$\frac{1}{2}\frac{d}{dt}\int\{J^s u J^s u + J^s\mathcal{H}u J^s\mathcal{H}u\}\phi. \tag{4.30}$$

The last two terms in (4.29) can be bounded by

$$\Big|\int\{J^s u J^s\partial_x^{-1}u - J^s u J^s\mathcal{H}\partial_x^{-1}u\}\phi(x)\Big| \leq c\,\|u(t)\|_s\|\partial_x^{-1}u(t)\|_s. \tag{4.31}$$

Integrating by parts we have that the third and fourth terms satisfy

$$\int \{-J^s\mathcal{H}\partial_x^2 uJ^s u + J^s\partial_x^2 uJ^s\mathcal{H}u\}\phi = \int J^s IuJ^s u\phi' - \int J^s I\mathcal{H}uJ^s\mathcal{H}u\phi'$$

where $If = (|\xi|\widehat{f}(\xi))^{\vee}$. Since the terms on the right hand side are similar we will estimate only the first term in the above expression. Observe that

$$\int J^s IuJ^s u\phi' = \int J^{s+1}uJ^s\phi' + \int J^s(I-J)uJ^s u\phi'$$
$$= \int J^{s+1/2}uJ^{s+1/2}u\phi' + \int J^{s+1/2}u[J^{1/2},\phi']J^s u + \int J^s(I-J)uJ^s u\phi'$$

where we used Plancherel's theorem. Since $\widehat{(I-J)}f(\xi) = m(\xi)\widehat{f}(\xi)$ with $m \in L^\infty$, Mihlin's theorem implies that

$$\Big|\int J^s(I-J)uJ^s\phi'\Big| \le c\|u(t)\|_s^2. \tag{4.32}$$

Also

$$\int J^{s+1/2}u[J^{1/2},\phi']J^s u = \int J^s u([J,\phi']J^s u - [J^{1/2},\phi']J^{s+1/2}u).$$

Lemma 2.1 in [15] implies that

$$\Big|\int J^{s+1/2}u[J^{1/2},\phi']J^s u\Big| \le c\|u(t)\|_s^2. \tag{4.33}$$

Next we estimate the term $\int J^s(u\partial_x u)J^s u\phi$. We rewrite it as

$$\int J^s(u\partial_x u)J^s u\phi = \int uJ^s\partial_x uJ^s u\phi + \int [J^s,u]\partial_x uJ^s u\phi$$
$$= -\frac{1}{2}\int \partial_x uJ^s uJ^s u\phi - \frac{1}{2}\int uJ^s uJ^s u\phi' + \int [J^s,u]\partial_x uJ^s u\phi.$$

Using the Cauchy-Schwartz inequality and Lemma 4.1 we have that

$$\Big|\int J^s(u\partial_x u)J^s u\phi\Big| \le c\|u(t)\|_{1,\infty}\|u(s)\|_s^2. \tag{4.34}$$

Next we estimate the term $\int J^s\mathcal{H}(u\partial_x u)J^s\mathcal{H}u\phi$. So we rewrite it and integrate it by parts to obtain

$$\int J^s\mathcal{H}(u\partial_x u)J^s\mathcal{H}u\phi = \int (J^s\mathcal{H}(u\partial_x u) - uJ^s\mathcal{H}\partial_x u)J^s\mathcal{H}u\phi + \int uJ^s\mathcal{H}\partial_x uJ^s\mathcal{H}u\phi$$
$$= \int \{\frac{1}{2}J^s(I-J)(u^2) + \frac{1}{2}J^{s+1}(u^2) - uJ^{s+1}u - uJ^s(I-J)u\}J^s\mathcal{H}u\phi$$
$$-\frac{1}{2}\int \partial_x uJ^s\mathcal{H}uJ^s\mathcal{H}u\phi - \frac{1}{2}\int uJ^s\mathcal{H}uJ^s\mathcal{H}u\phi'.$$

Hence applying Lemma 4.1 and Mihlin's theorem we get

$$\Big|\int J^s\mathcal{H}(u\partial_x u)J^s\mathcal{H}u\phi\Big| \le c\|u(t)\|_s^3 + \|J^{s+1}(u^2) - 2uJ^{s+1}u\|_{L^2}\|u(t)\|_{L^2} + \|u(t)\|_{L^\infty}\|u(t)\|_s^2 + \|u(t)\|_{1,\infty}\|u(t)\|_s^2. \tag{4.35}$$

To conclude the estimate we use Lemma 2.4 in [15] to lead to

$$\|J^{s+1}(u^2) - 2uJ^{s+1}u\|_{L^2} \le c\|u(t)\|_s^2, \qquad \text{when } s > 3/2. \tag{4.36}$$

Gathering together the estimates (4.30)–(4.36) and integration in time we have that

$$\int_0^T\int J^{s+1/2}uJ^{s+1/2}u\phi \le \|u_0\|_s^2 + \|u(t)\|_s^2 + c\int_0^T \|u(t')\|_{1,\infty}\|u(t)\|_s^2 + c\int_0^T \|u(t')\|_s^3\,dt' + c\,T\|u(t)\|_s^2\|\partial_x^{-1}u(t)\|_s.$$

Using the *a priori* estimate (4.26) we have

$$\int_0^T\int J^{s+1/2}uJ^{s+1/2}u\phi \le M(\|u_0\|_{X_s}, T, \phi).$$

Lemma 2.1 in [15] implies then that for $\phi \in C_0^\infty(\mathbb{R})$

$$J^{s+1/2}(u\phi) \in L^2([0,T] : L^2(\mathbb{R})).$$

This completes the proof.

□

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IMPA
Estrada Dona Castorina 110
22460-320, Rio de Janeiro, RJ, Brazil
Email: linares@impa.br

Universidade Federal de Minas Gerais
Instituto de Ciências Exatas
31270-901, Belo Horizonte, MG Brazil