GENERALIZED SOLUTIONS TO NONLINEAR WAVE EQUATIONS

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Dedicated to the memory of Professor Hebe de Azevedo Biagioni

Abstract

Starting from early results of Biagioni on semilinear hyperbolic equations we investigate solutions to the one-dimensional semilinear wave equation with generalized function data and singular potential in the Colombeau algebra of generalized functions. We prove an existence-uniqueness result, compute the associated distribution for a delta function potential in the nonlinear case and develop concepts of regularity of generalized solutions applicable in a nonlinear setting.

1 Introduction

In one of her early papers [2] on Colombeau theory, Biagioni proved that the initial value problem for a hyperbolic $(n \times n)$-system in one space variable

$$\partial_t u_i + \lambda_i(x,t) u_i = f_i(x,t,u_1,\ldots,u_n), \quad 1 \le i \le n \tag{1}$$

with initial data belonging to the Colombeau algebra $\mathcal{G}(I)$ on an interval $I \subset \mathbb{R}$ has a unique generalized solution in the sense of Colombeau on the corresponding domain of determinacy and that the solution depends continuously on the data. In this paper, Biagioni assumed that the coefficients λ_i are smooth and real valued and that the functions f_i are smooth, of polynomial growth and

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with all derivatives with respect to the variables u_i bounded. She also showed that the generalized solution is associated with the classical solution in case the initial data are continuously differentiable functions. This result, taken up in her monograph [3], was one of the early results establishing Colombeau theory as a tool for treating nonlinear partial differential equations with non-smooth data for which the classical theory would not provide any solution concept. It exhibits the main type of questions that have been posed and answered for a wealth of linear and nonlinear partial differential equations in the decades to follow:

(a) Existence and uniqueness of generalized solutions in Colombeau algebras;
(b) limiting behavior of the representatives when the data are distributions;
(c) regularity of generalized solutions.

Biagioni pursued this line of research for various classes of partial differential equations [5, 6, 7, 8, 9]. Concerning question (a), a short survey of later existence and uniqueness results can be found in [31]. For linear systems of type (1), existence and uniqueness was established for non-smooth (Colombeau generalized) coefficients λ_i in [28], for symmetric hyperbolic systems in higher space dimensions in [23] and for hyperbolic pseudodifferential systems with Colombeau symbols in [18]. Notably for semilinear hyperbolic systems as (1), question (b) has been answered in many cases, involving the notion of delta-waves [12, 16, 26, 29, 35, 36, 38]. Finally, regularity theory for Colombeau solutions is now based on the subalgebra $\mathcal{G}^\infty$ of regular Colombeau functions and currently an active area of research, making use of pseudodifferential and microlocal techniques [13, 14, 20, 24]. In particular, the propagation of the $\mathcal{G}^\infty$-wave front set in linear systems of type (1) with Colombeau coefficients is a theme of recent investigations [15]. To date, very little is known about regularity of Colombeau solutions in the nonlinear case.

The purpose of this note is to demonstrate recent methods for treating questions (a) - (c) in the case of a model equation, the semilinear wave equation with singular potential as well as singular driving term and initial data

$$\begin{aligned} &\partial_t^2 u(x,t) - \partial_x^2 u(x,t) = f(u(x,t))g(x) + h(x,t), \quad x \in \mathbb{R}, t > 0, \\ &u(x,0) = a(x),\ \partial_t u(x,0) = b(x), \quad x \in \mathbb{R}, \end{aligned} \tag{2}$$

where f is a smooth, polynomially bounded function and a, b, g and h are Colombeau generalized functions on the real line and on the upper half plane, respectively. After recalling the required notions from Colombeau theory in Section 2, we shall prove existence and uniqueness of a solution u belonging to the Colombeau algebra on the upper half plane in Section 3. Section 4 is devoted to computing the associated distribution (the distributional limit of the representing nets) when the potential g is a delta function. Both the existence and uniqueness result and the limiting result is new in case of a singular potential. In Section 5 we turn to regularity theory. We briefly recall the $\mathcal{G}^\infty$-regularity result for the linear case, show that it fails in the nonlinear case and finally prove a regularity result using so-called slow scale generalized functions in the nonlinear case. This result is new and based on the notion of a slow scale net which has been introduced in [14, 21] and has turned out to be central to regularity theory in the Colombeau setting.

For simplicity of presentation, we restrict our attention to the one-dimensional case and Lipschitz-continuous nonlinearity f. At the appropriate places of the paper, we will indicate what is known about the non-Lipschitz and the higher dimensional case. With Lipschitz-continuous f, in particular, the solution to (2) exists globally in space and time. Therefore, we need not enter the discussion of domains of existence. This latter question of the existence of local solutions is difficult to handle in the Colombeau setting, due to the fact that the time of existence may shrink to zero as the regularization parameter approaches zero. This led Biagioni to introduce the notion of a germ of generalized functions in [4], where she successfully solved nonlinear first order partial differential equations in an algebra of germs. This pioneering work awaits to be taken up and continued.

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2 Notation

The paper is placed in the framework of algebras of generalized functions introduced by Colombeau in [10, 11]. We shall fix the notation and introduce a number of known as well as new classes of generalized functions here. For more details, see [17].

Let Ω be an open subset of $\mathbb{R}^n$. The basic objects of the theory as we use it are families $(u_\varepsilon)_{\varepsilon\in(0,1]}$ of smooth functions $u_\varepsilon \in \mathcal{C}^\infty(\Omega)$ for $0 < \varepsilon \le 1$. We single out the following subalgebras:
Moderate families, denoted by $\mathcal{E}_{\mathrm{M}}(\Omega)$, are defined by the property:

$$\forall K \Subset \Omega\,\forall \alpha \in \mathbb{N}_0^n\,\exists p \ge 0: \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{-p}) \text{ as } \varepsilon \to 0. \tag{3}$$

Null families, denoted by $\mathcal{N}(\Omega)$, are defined by the property:

$$\forall K \Subset \Omega\,\forall \alpha \in \mathbb{N}_0^n\,\forall q \ge 0: \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{q}) \text{ as } \varepsilon \to 0. \tag{4}$$

Thus moderate families satisfy a locally uniform polynomial estimate as $\varepsilon \to 0$, together with all derivatives, while null functionals vanish faster than any power of ε in the same situation. The null families form a differential ideal in the collection of moderate families. The *Colombeau algebra* is the factor algebra

$$\mathcal{G}(\Omega) = \mathcal{E}_{\mathrm{M}}(\Omega)/\mathcal{N}(\Omega).$$

The algebra $\mathcal{G}(\Omega)$ just defined coincides with the *special Colombeau algebra* in [17, Def. 1.2.2], where the notation $\mathcal{G}^s(\Omega)$ has been employed. It was called the *simplified Colombeau algebra* in [3].

The Colombeau algebra on a closed half space $\mathbb{R}^n \times [0,\infty)$ is defined in a similar way. The restriction of an element $u \in \mathcal{G}(\mathbb{R}^n \times [0,\infty))$ to the line $\{t = 0\}$ is defined on representatives by

$$u|_{\{t=0\}} = \text{ class of } (u_\varepsilon(\cdot,0))_{\varepsilon\in(0,1]}.$$

Similarly, restrictions of the elements of $\mathcal{G}(\Omega)$ to open subsets of Ω are defined on representatives. One can see that $\Omega \to \mathcal{G}(\Omega)$ is a sheaf of differential algebras on $\mathbb{R}^n$. The space of compactly supported distributions is imbedded in $\mathcal{G}(\Omega)$ by convolution:

$$\iota : \mathcal{E}'(\Omega) \to \mathcal{G}(\Omega),\ \iota(w) = \text{ class of } (w * (\varphi_\varepsilon)|_\Omega)_{\varepsilon\in(0,1]}, \tag{5}$$

where

$$\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi\left(x/\varepsilon\right) \tag{6}$$

is obtained by scaling a fixed test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ of integral one with all moments vanishing. By the sheaf property, this can be extended in a unique way to an imbedding of the space of distributions $\mathcal{D}'(\Omega)$.

One of the main features of the Colombeau construction is the fact that this imbedding renders $\mathcal{C}^\infty(\Omega)$ a faithful subalgebra. In fact, given $f \in \mathcal{C}^\infty(\Omega)$, one can define a corresponding element of $\mathcal{G}(\Omega)$ by the constant imbedding $\sigma(f) =$ class of $[(\varepsilon, x) \to f(x)]$. Then the important equality $\iota(f) = \sigma(f)$ holds in $\mathcal{G}(\Omega)$.

If $u \in \mathcal{G}(\Omega)$ and f is a smooth function which is of at most polynomial growth at infinity, together with all its derivatives, the superposition $f(u)$ is a well-defined element of $\mathcal{G}(\Omega)$.

We need a couple of further notions from the theory of Colombeau generalized functions. An element u of $\mathcal{G}(\Omega)$ is called of *local* L^p*-type* $(1 \le p \le \infty)$, if it has a representative with the property

$$\limsup_{\varepsilon\to 0} \|u_\varepsilon\|_{\mathrm{L}^p(K)} < \infty$$

for every $K \Subset \Omega$.

Regularity theory is based on the subalgebra $\mathcal{G}^\infty(\Omega)$ of *regular generalized functions* in $\mathcal{G}(\Omega)$. It is defined by those elements which have a representative satisfying

$$\forall K \Subset \Omega\, \exists p \ge 0\, \forall \alpha \in \mathbb{N}_0^n : \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{-p}) \ \text{ as } \varepsilon \to 0. \tag{7}$$

Observe the change of quantifiers with respect to formula (3); locally, all derivatives of a regular generalized function have the same order of growth in $\varepsilon > 0$. One has that (see [30, Thm. 5.2])

$$\mathcal{G}^\infty(\Omega) \cap \mathcal{D}'(\Omega) = \mathcal{C}^\infty(\Omega).$$

For the purpose of describing the regularity of Colombeau generalized functions, $\mathcal{G}^\infty(\Omega)$ plays the same role as $\mathcal{C}^\infty(\Omega)$ does in the setting of distributions.

A net $(r_\varepsilon)_{\varepsilon\in(0,1]}$ of complex numbers is called a *slow scale net* if

$$|r_\varepsilon|^p = \mathcal{O}(\varepsilon^{-1}) \text{ as } \varepsilon \to 0$$

for every $p \geq 0$. We refer to [21] for a detailed discussion of slow scale nets. Finally, an element $u \in \mathcal{G}(\Omega)$ is called of *total slow scale type*, if for some representative, $\|\partial^\alpha u_\varepsilon\|_{\mathrm{L}^\infty(K)}$ forms a slow scale net for every $K \Subset \Omega$ and $\alpha \in \mathbb{N}_0^n$.

It is clear that every element u of $\mathcal{G}(\Omega)$ which is of total slow scale type belongs to $\mathcal{G}^\infty(\Omega)$ (but not conversely). However, in the opposite direction estimates on the zero-th derivatives suffice. More precisely, if u belongs to $\mathcal{G}^\infty(\Omega)$ and is of local L^∞-type, then it is already of total slow scale type (see [22]).

We end this section by recalling the *association relation* on the Colombeau algebra $\mathcal{G}(\Omega)$. It identifies elements of $\mathcal{G}(\Omega)$ if they coincide in the weak limit. That is, $u, v \in \mathcal{G}(\Omega)$ are called associated, $u \approx v$, if $\lim_{\varepsilon\to 0} \int \big(u_\varepsilon(x) - v_\varepsilon(x)\big)\psi(x)\,dx = 0$ for all test functions $\psi \in \mathcal{D}(\Omega)$. We shall also say that u is associated with a distribution w if $u_\varepsilon \to w$ in the sense of distributions as $\varepsilon \to 0$.

3 Existence/uniqueness of generalized solutions

This section is devoted to solving the semilinear wave equation (2) in the Colombeau algebra $\mathcal{G}(\mathbb{R} \times [0,\infty))$. Recall first that if w is a classical solution of the linear wave equation

$$\begin{aligned} &\partial_t^2 w(x,t) - \partial_x^2 w(x,t) = h(x,t), \quad x \in \mathbb{R}, t > 0,\\ &w(x,0) = a(x),\ \partial_t w(x,0) = b(x), \quad x \in \mathbb{R}, \end{aligned} \tag{8}$$

then it solves the integral equation

$$w(x,t) = \frac{1}{2}\big(a(x-t)+a(x+t)\big)+\frac{1}{2}\int_{x-t}^{x+t} b(y)\,\mathrm{d}y+\frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} h(y,s)\,\mathrm{d}y\,\mathrm{d}s. \quad (9)$$

Let $I_0 = [-\kappa, \kappa]$ be a compact interval. For $0 \leq t \leq s \leq \kappa$, the interval I_t and the trapezoidal region K_s are defined by

$$\begin{array}{lcl} I_t & = & \{x \in \mathbb{R} : |x| \leq \kappa - t\}, \\ K_s & = & \{(x,t) \in \mathbb{R} \times [0,\infty) : 0 \leq t \leq s, x \in I_t\}. \end{array} \quad (10)$$

Using (9), the following estimates are easily deduced ($0 \leq t \leq T \leq \kappa$):

$$\|w\|_{\mathrm{L}^\infty(K_T)} \leq \|a\|_{\mathrm{L}^\infty(I_0)} + T\|b\|_{\mathrm{L}^\infty(I_0)} + T\int_0^T \|h\|_{\mathrm{L}^\infty(K_s)}\,\mathrm{d}s, \quad (11)$$

$$\|w(\cdot,t)\|_{\mathrm{L}^\infty(I_t)} \leq \|a\|_{\mathrm{L}^\infty(I_0)} + T\|b\|_{\mathrm{L}^\infty(I_0)} + \frac{1}{2}\int_0^t \|h(\cdot,s)\|_{\mathrm{L}^1(I_s)}\,\mathrm{d}s. \quad (12)$$

We now turn to the semilinear wave equation (2). We assume throughout that $u \to f(u)$ is a smooth function all whose derivatives are of at most polynomial growth as $|u| \to \infty$, that f satisfies a global Lipschitz estimate (i.e., has a bounded first derivative) and that $f(0) = 0$.

Proposition 1. *Assume that the function f is as described above. Let $a, b \in \mathcal{G}(\mathbb{R})$, $h \in \mathcal{G}(\mathbb{R} \times [0,\infty))$ and let $g \in \mathcal{G}(\mathbb{R})$ be of local* L^1*-type. Then problem (2) has a unique solution $u \in \mathcal{G}(\mathbb{R} \times [0,\infty))$.*

Proof: To prove the existence of a solution, take representatives $a_\varepsilon, b_\varepsilon, g_\varepsilon, h_\varepsilon$ of a, b, g, h, respectively, and let $u_\varepsilon \in \mathcal{C}^\infty(\mathbb{R} \times [0,\infty))$ be the unique solution to the semilinear wave equation with regularized data:

$$\begin{array}{ll} \partial_t^2 u_\varepsilon - \partial_x^2 u_\varepsilon = f(u_\varepsilon)\,g_\varepsilon + h_\varepsilon & \text{on } \mathbb{R} \times [0,\infty), \\ u_\varepsilon(\cdot,0) = a_\varepsilon,\ \partial_t u_\varepsilon(\cdot,0) = b_\varepsilon & \text{on } \mathbb{R}. \end{array} \quad (13)$$

The classical solution u_ε to (13) is constructed by rewriting (13) as an integral equation and invoking a fixed point argument (this involves applying estimate (11) successively to all derivatives). If we show that the net $(u_\varepsilon)_{\varepsilon\in(0,1]}$ belongs to $\mathcal{E}_{\mathrm{M}}(\mathbb{R} \times [0,\infty))$, its equivalence class in $\mathcal{G}(\mathbb{R} \times [0,\infty))$ will be a solution. To show that the zero-th derivative of u_ε satisfies the estimate (3), we take a region K_T with its horizontal slices I_t and invoke inequality (12) to see that

$$\begin{aligned} \|u_\varepsilon(\cdot,t)\|_{\mathrm{L}^\infty(I_t)} \leq \; & \|a_\varepsilon\|_{\mathrm{L}^\infty(I_0)} + T\|b_\varepsilon\|_{\mathrm{L}^\infty(I_0)} \\ & + \frac{1}{2}\int_0^t \|f(u_\varepsilon(\cdot,s))g_\varepsilon + h_\varepsilon(\cdot,s)\|_{\mathrm{L}^1(I_s)}\,\mathrm{d}s. \end{aligned} \tag{14}$$

The last term on the right hand side of (14) is estimated by

$$\frac{1}{2}\int_0^t \big(\|f'\|_{\mathrm{L}^\infty(\mathbb{R})}\|(u_\varepsilon(\cdot,s))\|_{\mathrm{L}^\infty(I_s)}\|g_\varepsilon\|_{\mathrm{L}^1(I_s)} + 2s\|h_\varepsilon(\cdot,s)\|_{\mathrm{L}^\infty(I_s)}\big)\,\mathrm{d}s. \tag{15}$$

Using that g is of local L^1-type and each of the terms involving $a_\varepsilon, b_\varepsilon, h_\varepsilon$ is of order $\mathcal{O}(\varepsilon^{-p})$ for some p, we infer from Gronwall's inequality that the same is true of $\|u_\varepsilon(\cdot,t)\|_{\mathrm{L}^\infty(I_t)}$ for $0 \leq t \leq T$. Thus u_ε is moderate on the region K_T, that is, it satisfies the estimate (3) there. To get the estimates for the higher order derivatives, one just differentiates the equation and employs the same arguments inductively, using that the lower order terms are already known to be moderate from the previous steps.

To prove uniqueness, we consider representatives $u_\varepsilon, v_\varepsilon \in \mathcal{E}_\mathrm{M}[\mathbb{R}\times[0,\infty))$ of two solutions u and v. Their difference satisfies

$$\begin{aligned} &\partial_t^2(u_\varepsilon - v_\varepsilon) - \partial_x^2(u_\varepsilon - v_\varepsilon) = (f(u_\varepsilon) - f(v_\varepsilon))\,g_\varepsilon + n_\varepsilon, \\ &(u_\varepsilon(\cdot,0) - v_\varepsilon(\cdot,0)) = n_{0\varepsilon},\ \partial_t(u_\varepsilon(\cdot,0) - v_\varepsilon(\cdot,0)) = n_{1\varepsilon} \end{aligned}$$

for certain null elements $n_\varepsilon, n_{0\varepsilon}, n_{1\varepsilon}$. Thus $u_\varepsilon - v_\varepsilon$ satisfies an estimate of the form (14), but with the null elements $n_\varepsilon, n_{0\varepsilon}, n_{1\varepsilon}$ replacing $a_\varepsilon, b_\varepsilon, h_\varepsilon$ there. This implies as above that the L^∞-norm of $u_\varepsilon - v_\varepsilon$ on K_T is of order $\mathcal{O}(\varepsilon^q)$ for every $q \geq 0$. By [17, Thm. 1.2.3], the null estimate (4) on $u_\varepsilon - v_\varepsilon$ suffices to have null estimates on all derivatives. Thus $u = v$ in $\mathcal{G}(\mathbb{R}\times[0,\infty))$.

□

Remark 2. *In case the data are continuous or smooth functions, the relation of the generalized solution to the classical solution is as follows. Assume first that* a, b, g *belong to* $\mathcal{C}^\infty(\mathbb{R})$ *and* h *to* $\mathcal{C}^\infty(\mathbb{R}\times[0,\infty))$. *Let* $w \in \mathcal{C}^\infty(\mathbb{R}\times[0,\infty))$ *be the classical solution. Then* w *coincides with the generalized solution* $u \in \mathcal{G}(\mathbb{R}\times[0,\infty))$, *that is,* $u = \iota(w)$ *in* $\mathcal{G}(\mathbb{R}\times[0,\infty))$. *This follows from the* *fact that the imbedding ι coincides with the constant imbedding σ on $\mathcal{C}^\infty(\mathbb{R})$, so $u_\varepsilon \equiv w$ is a representative of the generalized solution. Second, assume the data a, b, g and h are continuous functions and let $w \in \mathcal{C}(\mathbb{R} \times [0,\infty))$ be the corresponding continuous (weak) solution. Then the generalized solution $u \in \mathcal{G}(\mathbb{R} \times [0,\infty))$ is associated with w. This follows from the classical result of continuous dependence of the continuous solution on the initial data. Third, when the data are distributions, there may be no meaning for a distributional solution, in general. Yet the solution in $\mathcal{G}(\mathbb{R} \times [0,\infty))$ may still be associated with a distribution. Some incidents of such a situation will be described in Section 4.*

Proposition 1 is a model result. In fact, for the case without potential and driving term ($g \equiv 1, h \equiv 0$), it is a special case of Biagioni's paper [2] and of [27]; for Lipschitz continuous, smooth f and $g \equiv 1$, existence and uniqueness of a solution in $\mathcal{G}(\mathbb{R}^n \times [0,\infty))$ can be proven in space dimensions $n = 1, 2, 3$, see e.g. [31]. If f is not Lipschitz, but of polynomial growth, energy estimates can be used to construct solutions in the Colombeau algebra $\mathcal{G}_{2,2}(\mathbb{R}^n \times [0,\infty))$ introduced in [9]. As in the classical case, the growth type of f is connected with the space dimension; the cases $1 \le n \le 6$ have been treated in [25]; for $n = 3$ see also [11].

4 Singular potentials

In this section we investigate the behavior of the solution $u \in \mathcal{G}(\mathbb{R} \times [0,\infty))$ to the semilinear wave equation (2) when the potential g is given by a measure. As a model situation we consider a delta function potential, that is, a potential given by $\iota(\delta)$ where $\delta \in \mathcal{D}'(\mathbb{R})$ is the Dirac measure. According to (5) and (6), a representative of $\iota(\delta)$ is given by the net $(\varphi_\varepsilon)_{\varepsilon\in(0,1]}$. Thus we consider the problem

$$\begin{aligned} &\partial_t^2 u - \partial_x^2 u = f(u)\iota(\delta) && \text{on} \quad \mathbb{R} \times [0,\infty), \\ &u(\cdot,0) = \iota(a),\ \partial_t u(\cdot,0) = \iota(b) && \text{on} \quad \mathbb{R}. \end{aligned} \tag{16}$$

We are not concerned with propagation of strong singularities from the initial data, so we will assume that a and b belong $\mathcal{C}_\mathrm{b}(\mathbb{R})$, the space of bounded and continuous functions. In this case, there is a solution concept for the corresponding classical equation

$$\begin{array}{lll} \partial_t^2 w - \partial_x^2 w = f(w)\,\delta & \text{on} & \mathbb{R}\times[0,\infty), \\ w(\cdot,0) = a,\ \partial_t w(\cdot,0) = b & \text{on} & \mathbb{R} \end{array} \tag{17}$$

when w is a continuous function on $\mathbb{R}\times[0,\infty)$: the derivatives on the left hand side may be interpreted in the weak sense, the product on the right hand side as the multiplication of a continuous function and a measure. We are going to show that equation (17) has a unique continuous weak solution w and that the generalized solution u to (16) is associated with it. The assumptions on the function f are the same as in Section 3.

The fundamental solution for the linear wave equation on $\mathbb{R}\times[0,\infty)$ is given by

$$E(t,x) = \frac{1}{2}\mathrm{H}(t-|x|)$$

where H denotes the Heaviside function. Fix $T>0$ and some bounded and continuous function ϕ on $\mathbb{R}\times[0,T]$. The linear equation

$$\begin{array}{lll} \partial_t^2 v - \partial_x^2 v = \phi\delta & \text{on} & \mathbb{R}\times[0,T], \\ v(\cdot,0) = 0,\ \partial_t v(\cdot,0) = 0 & \text{on} & \mathbb{R} \end{array} \tag{18}$$

has the weak solution

$$\begin{aligned} v(x,t) &= \left(\int_0^t E(t-s,\cdot) * \phi(0,s)\delta\,\mathrm{d}s\right)(x) \\ &= \frac{1}{2}\int_0^t \mathrm{H}(t-s-|x|)\phi(0,s)\,\mathrm{d}s \\ &= \frac{1}{2}\mathrm{H}(t-|x|)\left(\mathrm{H}(-x)\int_0^{t+x}\phi(0,s)\,\mathrm{d}s + \mathrm{H}(x)\int_0^{t-x}\phi(0,s)\,\mathrm{d}s\right) \\ &= \frac{1}{2}\mathrm{H}(t-|x|)\int_0^{t-|x|}\phi(0,s)\,\mathrm{d}s. \end{aligned}$$

The explicit representation shows that v is continuous, its support intersected with the strip $\mathbb{R}\times[0,T]$ is compact and the following estimate holds:

$$\|v\|_{\mathrm{L}^\infty(\mathbb{R}\times[0,T])} \le T\|\phi\|_{\mathrm{L}^\infty(\mathbb{R}\times[0,T])}.$$

Lemma 3. *Assume that f is a globally Lipschitz continuous function and that a, b belong to $\mathcal{C}_{\mathrm{b}}(\mathbb{R})$. Then equation (17) has a unique solution $w \in \mathcal{C}(\mathbb{R} \times [0,\infty)) \cap \mathcal{C}^1([0,\infty) : \mathcal{D}'(\mathbb{R}))$ in the sense described above. In addition, $w \in \mathcal{C}_{\mathrm{b}}(\mathbb{R} \times [0,T])$ for every $T > 0$.*

Proof: By the explicit calculation for the linear equation (18), it is clear that - for solutions with the property stated in the lemma - equation (17) is equivalent with the integral equation

$$\begin{aligned} w(x,t) &= \frac{1}{2}\big(a(x-t) + a(x+t)\big) + \frac{1}{2}\int_{x-t}^{x+t} b(y)\,\mathrm{d}y \\ &\quad + \frac{1}{2}\mathrm{H}(t-|x|)\int_0^{t-|x|} f(w(0,s))\,\mathrm{d}s. \end{aligned} \tag{19}$$

For small T, the right hand side of this integral equation defines a contraction in a ball in $\mathcal{C}_{\mathrm{b}}(\mathbb{R} \times [0,T])$ around the solution of the homogeneous linear wave equation. The time T of existence of the local solution thus constructed depends only on the size of the initial data and the Lipschitz constant of f, thus the solution exists globally in time. Uniqueness follows by applying Gronwall's lemma to the integral equation satisfied by $w(0,t)$ (see the remark below).

□

Remark 4. *The classical weak solution can be calculated more explicitly. Indeed, inserting $x = 0$ in (19) and differentiating with respect to t leads to the ordinary differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}w(0,t) = \frac{1}{2}\big(a'(t) - a'(-t) + b(t) + b(-t) + f(w(0,t))\big)$$

with initial value $w(0,0) = a(0)$. Knowledge of $w(0,t)$ in turn allows to compute $w(x,t)$ by means of (19).

Proposition 5. *Under the assumptions on a, b and f above, the generalized solution $u \in \mathcal{G}(\mathbb{R} \times [0,\infty))$ to (16) constructed in Prop. 1 is associated with the solution $w \in \mathcal{C}(\mathbb{R} \times [0,\infty))$ to (17) given by Lemma 3.*

Proof: The solution $u \in \mathcal{G}(\mathbb{R} \times [0,\infty))$ has a representative which satisfies

$$\begin{array}{ll} \partial_t^2 u_\varepsilon - \partial_x^2 u_\varepsilon = f(u_\varepsilon)\varphi_\varepsilon & \text{on} \quad \mathbb{R} \times [0,\infty), \\ u_\varepsilon(\cdot,0) = a_\varepsilon,\ \partial_t u_\varepsilon(\cdot,0) = b_\varepsilon & \text{on} \quad \mathbb{R} \end{array}$$

where $a_\varepsilon = a * \varphi_\varepsilon, b_\varepsilon = b * \varphi_\varepsilon$. Let

$$v(x,t) = \frac{1}{2}\mathrm{H}(t-|x|)\int_0^{t-|x|} f(w(0,s))\,\mathrm{d}s$$

and $v_\varepsilon = v * \varphi_\varepsilon$ (convolution in the space variable only), so that

$$\begin{array}{rcl} \partial_t^2 v(x,t) - \partial_x^2 v(x,t) & = & f(w(0,t))\delta(x), \\ \partial_t^2 v_\varepsilon(x,t) - \partial_x^2 v_\varepsilon(x,t) & = & f(w(0,t))\varphi_\varepsilon(x) \end{array}$$

with vanishing initial data. It is clear from its definition that v is continuous with compact support in any strip $\mathbb{R} \times [0,T]$. Thus v is uniformly continuous and

$$\lim_{\varepsilon\to 0} \|v - v_\varepsilon\|_{\mathrm{L}^\infty(\mathbb{R}\times[0,T])} = 0.$$

We have that

$$\left(\partial_t^2 - \partial_x^2\right)(u_\varepsilon - v_\varepsilon - w + v) = \big(f(u_\varepsilon) - f(w)\big)\varphi_\varepsilon + \big(f(w) - f(w(0,\cdot))\big)\varphi_\varepsilon$$

with initial data

$$(u_\varepsilon - v_\varepsilon - w + v)|_{\{t=0\}} = a_\varepsilon - a,\ \partial_t(u_\varepsilon - v_\varepsilon - w + v)|_{\{t=0\}} = b_\varepsilon - b.$$

Consider a region K_T with its horizontal slices I_t as in (10). Estimate (12) gives

$$\begin{aligned} \|u_\varepsilon - w\|_{\mathrm{L}^\infty(I_t)} - \|v_\varepsilon - v\|_{\mathrm{L}^\infty(I_t)} &\le \|u_\varepsilon - v_\varepsilon - w + v\|_{\mathrm{L}^\infty(I_t)} \\ &\le \frac{1}{2}\int_0^t \|f'\|_{\mathrm{L}^\infty(\mathbb{R})}\|u_\varepsilon - w\|_{\mathrm{L}^\infty(I_s)}\|\varphi_\varepsilon\|_{\mathrm{L}^1(I_s)}\,\mathrm{d}s \\ &+ \frac{1}{2}\int_0^t \|\big(f(w(\cdot,s)) - f(w(0,s))\big)\varphi_\varepsilon\|_{\mathrm{L}^1(I_s)}\,\mathrm{d}s \\ &\qquad + \|a_\varepsilon - a\|_{\mathrm{L}^\infty(I_0)} + T\|b_\varepsilon - b\|_{\mathrm{L}^\infty(I_0)}. \end{aligned}$$

Now $\|\varphi_\varepsilon\|_{\mathrm{L}^1(\mathbb{R})} = 1$, its mass accumulates at $\{0\}$ as $\varepsilon \to 0$ and w is uniformly continuous on K_T. Thus the second to last term on the right hand side goes to zero, as do the terms involving $v_\varepsilon - v$ and $a_\varepsilon - a$, $b_\varepsilon - b$. Applying Gronwall's inequality we conclude that

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - w\|_{\mathrm{L}^\infty(I_t)} = 0,$$

uniformly for $t \in [0, T]$. In particular, u_ε converges to w in the sense of distributions; this means that u is associated with w.

□

The result can be generalized to measures as potentials and to more singular initial data. Associated distributions to the generalized solution of the semilinear wave equation (2) without potential ($g \equiv 1$) have been obtained earlier. In case the initial data a, b and the driving term h are distributions with discrete support, such associated distributions are termed delta-waves. Delta waves have been shown to exist for bounded or sublinear f in space dimensions $n = 1, 2, 3$ e.g. in [26, 29, 35]. In case the driving term is white noise, the generalized solution to the semilinear (stochastic) wave equation is associated with the solution of a linear stochastic wave equation in many cases. This has been shown e.g. in [1, 32, 34]. White noise or positive noise in the initial data has been considered in [33, 37].

5 Nonlinear regularity theory

Let $w \in \mathcal{G}(\mathbb{R} \times [0, \infty))$ be a solution to the linear wave equation (8) with $a, b \in \mathcal{G}(\mathbb{R})$, $h \in \mathcal{G}(\mathbb{R} \times [0, \infty))$. It is clear from Remark 2 that if a, b and h are $\mathcal{C}^\infty$-functions, so is the solution w. However, the $\mathcal{C}^\infty$-category is not appropriate to describe propagation of singularities in the setting of Colombeau algebras. For example, the generalized solution to the linear wave equation with the square of the Dirac measure $\iota(\delta)^2$ as initial data is not $\mathcal{C}^\infty$-regular inside the light cone (see [31]), but it is $\mathcal{G}^\infty$-regular there. That the $\mathcal{G}^\infty$-category is appropriate for describing propagation of singularities for the linear wave equation in any space dimension has been shown in [30].

Remark 6. *A global $\mathcal{G}^\infty$-regularity result for the linear wave equation is easy to prove. In fact, let $w \in \mathcal{G}(\mathbb{R} \times [0, \infty))$ be the solution to the linear wave equation (8) with $\mathcal{G}^\infty$-regular data $a, b \in \mathcal{G}^\infty(\mathbb{R}), h \in \mathcal{G}^\infty(\mathbb{R} \times [0, \infty))$. Then w belongs to $\mathcal{G}^\infty(\mathbb{R} \times [0, \infty))$ as well. Indeed, estimate (11) applied to representatives shows that the representative w_ε of the solution inherits the order of growth in ε*

from the representatives $a_\varepsilon, b_\varepsilon, h_\varepsilon$. But the higher order derivatives of w_ε satisfy again the linear wave equation with correspondingly differentiated data. Since all derivatives of the data $a_\varepsilon, b_\varepsilon, h_\varepsilon$ have the same order of growth in ε, so do the derivatives of w_ε, that is, w satisfies (7) and so it belongs to $\mathcal{G}^\infty(\mathbb{R}\times[0,\infty))$.

The situation is different in the nonlinear case. To be sure, $\mathcal{G}^\infty(\mathbb{R})$ is not invariant under superposition with arbitrary nonlinear functions. For example, the net defined by $\chi_\varepsilon(x) = \frac{x}{\varepsilon}$ belongs to $\mathcal{E}^\infty_{\mathrm{M}}(\mathbb{R})$, but $\sin\frac{x}{\varepsilon}$ or $\cos\frac{x}{\varepsilon}$ do not. An alternative is offered by the subalgebra of $\mathcal{G}^\infty(\Omega)$ of elements of total slow scale type, introduced at the end Section 1.

Lemma 7. *Let Ω be an open subset of $\mathbb{R}^n$, $u \in \mathcal{G}(\Omega)$ of total slow scale type and f a smooth function all whose derivatives grow at most polynomially. Then $f(u)$ is of total slow scale type.*

Proof: Take a representative $(u_\varepsilon)_{\varepsilon\in(0,1]}$. Then

$$\big(\|\partial^\alpha u_\varepsilon\|_{\mathrm{L}^\infty(K)}\big)^p = \mathcal{O}(\varepsilon^{-1})$$

for every $K \Subset \Omega, \alpha \in \mathbb{N}_0^n$ and every $p \geq 0$. The polynomial bounds on f lead to the estimates

$$|f(u_\varepsilon)|^p \leq C_0\big(1+|u_\varepsilon|^{m_0}\big)^p, \quad |\partial_i(f(u_\varepsilon))|^p \leq C_i\big((1+|u_\varepsilon|^{m_i})|\partial_i u_\varepsilon|\big)^p,$$

and so on. This shows that all positive powers of all derivatives of $f(u_\varepsilon)$ are of order $\mathcal{O}(\varepsilon^{-1})$ on K.

□

We now turn the semilinear wave equation (2) and show that total slow scale regularity of the data is inherited by the solution. The assumptions on the function f are again the same as in Section 3.

Proposition 8. *Let $u \in \mathcal{G}(\mathbb{R}\times[0,\infty))$ be the solution to the semilinear wave equation (2) given in Prop. 1. Assume that the initial data a, b, the potential g and the driving term h are of total slow scale type and that g is of local L^1-type. Then u is of total slow scale type.*

Proof: Let K_T be a region as in the proof of Prop. 1. It follows from the estimates (14) and (15) and Gronwall's inequality that $\|u_\varepsilon\|_{\mathrm{L}^\infty(K_T)}$ can be bounded by a linear combination of $\|a_\varepsilon\|_{\mathrm{L}^\infty(I_0)}$, $\|b_\varepsilon\|_{\mathrm{L}^\infty(I_0)}$ and $\|h_\varepsilon\|_{\mathrm{L}^\infty(K_T)}$. This shows that every positive power of $\|u_\varepsilon\|_{\mathrm{L}^\infty(K_T)}$ is of order $\mathcal{O}(\varepsilon^{-1})$. The first and second derivatives of u_ε with respect to x satisfy

$$\begin{aligned}
\left(\partial_t^2 - \partial_x^2\right)(\partial_x u_\varepsilon) &= f'(u_\varepsilon)g_\varepsilon \partial_x u_\varepsilon + f(u_\varepsilon)\partial_x g_\varepsilon + \partial_x h_\varepsilon, \\
\left(\partial_t^2 - \partial_x^2\right)(\partial_x^2 u_\varepsilon) &= f'(u_\varepsilon)g_\varepsilon \partial_x^2 u_\varepsilon + f''(u_\varepsilon)g_\varepsilon(\partial_x u_\varepsilon)^2 \\
&\quad + 2f'(u_\varepsilon)\partial_x g_\varepsilon \partial_x u_\varepsilon + f(u_\varepsilon)\partial_x^2 g_\varepsilon + \partial_x^2 h_\varepsilon
\end{aligned}$$

with initial data

$$\begin{array}{l}
\partial_x u_\varepsilon|\{t=0\} = \partial_x a_\varepsilon, \quad \partial_t(\partial_x u_\varepsilon)|\{t=0\} = \partial_x b_\varepsilon, \\
\partial_x^2 u_\varepsilon|\{t=0\} = \partial_x^2 a_\varepsilon, \quad \partial_t(\partial_x^2 u_\varepsilon)|\{t=0\} = \partial_x^2 b_\varepsilon,
\end{array}$$

respectively. Using the same argument as above, one can estimate $\|\partial_x u_\varepsilon\|_{\mathrm{L}^\infty(K_T)}$ in terms of $\|\partial_x a_\varepsilon\|_{\mathrm{L}^\infty(I_0)}$, $\|\partial_x b_\varepsilon\|_{\mathrm{L}^\infty(I_0)}$ $\|\partial_x g_\varepsilon\|_{\mathrm{L}^\infty(K_T)}$, $\|\partial_x h_\varepsilon\|_{\mathrm{L}^\infty(K_T)}$ as well as by $\|f(u_\varepsilon)\|_{\mathrm{L}^\infty(K_T)}$, all whose powers are of order $\mathcal{O}(\varepsilon^{-1})$ by Lemma 7. The equation for $\partial_x^2 u_\varepsilon$ exhibits the same structure, so that a similar estimate holds for $\|\partial_x^2 u_\varepsilon\|_{\mathrm{L}^\infty(K_T)}$. ¿From here an inductive procedure yields the result.

□

Another promising approach to measuring regularity in the nonlinear case is the Colombeau-Hölder-Zygmund-scale which has been introduced and applied to nonlinear scalar first order equations in [19].

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