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THE PARALLEL DERIVATIVE

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Abstract

This paper concerns the definition and applications of the *parallel derivative* of a differentiable fiber bundle morphism between differentiable vector bundles endowed with connections. This is a dual concept to the well known *fiber derivative* of such morphisms, and the combined use of both fiber and parallel derivatives has proven to be a powerful tool to make coordinate-free computations.

1 Introduction

The aim of this paper is to introduce the "parallel derivative" of a differentiable fiber bundle morphism between vector bundles endowed with connections. This is a dual concept to the so-called "fiber derivative" of such morphisms. In [9], [8] and [7], the combined use of both derivatives was proven to be a powerful technique to make computations in a coordinate free manner.

The organization of the paper is the following: in section 2 we introduce some definitions and notation. In section 3 we define the parallel derivative and prove a theorem relating higher order fiber/parallel derivatives of a differentiable fiber bundle morphism between differentiable vector bundles endowed with connections to the curvature tensors of these connections. Finally, in section 4, we present some applications showing how these derivatives can be used to make intrinsic calculations.

2 Basic notations and definitions

In this section we set up some notation and basic definitions.

Keywords. Parallel derivative, fiber derivative, connections on vector bundles.

We denote by M a smooth connected paracompact finite-dimensional manifold; TM is the tangent bundle of M, and $\mathsf{T}^*\mathsf{M}$ its cotangent bundle, and we denote by $\tau_{\mathsf{M}} : \mathsf{TM} \to \mathsf{M}$, $\tau^*_{\mathsf{M}} : \mathsf{T}^*\mathsf{M} \to \mathsf{M}$ the corresponding projections. The trivial vector bundle over M with fiber F is denoted by F_{M}. By "differentiable" or "smooth" we mean C^∞. The set of differentiable functions on M, differentiable vector fields on M and differentiable forms on M are denoted by $\mathfrak{F}(\mathsf{M})$, $\mathfrak{X}(\mathsf{M})$ and $\Omega(\mathsf{M})$, respectively. If $\pi_E : E \to \mathsf{M}$ is a smooth vector bundle, we denote by 0_E (or simply 0) the zero section of E, i.e. $0_E = \{0_p : p \in \mathsf{M}\}$, where 0_p is the zero vector of $E_p = \pi_E^{-1}[p]$, $p \in \mathsf{M}$. The set of smooth sections of $\pi_E : E \to \mathsf{M}$ is denoted by $\Gamma^\infty(E)$.

In the sequel, we recall some notions regarding the geometry of tangent bundle $\mathsf{T}E$ of a smooth vector bundle E over M (see, for example, [4], [3], [6], [2], [5], [1]), which we shall use later on.

Given a smooth vector bundle $\pi_E : E \to \mathsf{M}$, the *vertical lift* λ^E is the smooth vector bundle morphism (where π_1 is the projection on the first factor):

$$\begin{tikzcd}
E \oplus_{\mathsf{M}} E \arrow[r, "\lambda^E"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathsf{T}E \arrow[d, "\tau_E"] \\
E \arrow[r, "\mathrm{id}_E"'] & E
\end{tikzcd}$$

such that, for all $q \in \mathsf{M}$, $u, v \in E_q$, $\lambda^E(u, v)$ is the image of v by the natural isomorphism $E_q \to \mathsf{T}_u(E_q)$ of the fiber E_q of $\pi_E : E \to \mathsf{M}$ over q with its tangent space at u, that is, $\lambda^E(u, v) = \frac{T}{dt}_{|t=0}(u + tv)$.

The image of λ^E is the *vertical sub-bundle* $\mathrm{Ver}(E) = \ker \mathsf{T}\pi_E$ of the tangent bundle of E. Since λ^E is a monomorphism, it is an isomorphism of smooth vector bundles onto $\mathrm{Ver}(E)$; we denote by $\widetilde{\kappa^V_E} : \mathrm{Ver}(E) \to E \oplus_{\mathsf{M}} E$ the inverse of λ^E, and by $\kappa^V_E : \mathrm{Ver}(E) \to E$ the composite $\pi_2 \circ \widetilde{\kappa^V_E}$, where π_2 is the projection on the second factor. Besides, for $v_q \in E$, we call $\lambda^E_{v_q} \doteq \lambda^E(v_q, \cdot) : E_q \to \mathrm{Ver}_{v_q}(E)$ the *vertical lift at* v_q, where $\mathrm{Ver}_{v_q}(E)$ is the fiber of $\mathrm{Ver}(E)$ over v_q.

An affine connection (or, simply, a connection) on $\pi_E : \mathsf{T}E \to \mathsf{M}$ is a smooth vector sub-bundle $\mathrm{Hor}(E)$ of $\mathsf{T}E$ satisfying the following conditions:

(∇1) $\mathsf{T}E = \mathrm{Hor}(E) \oplus_E \mathrm{Ver}(E)$, i.e. $\mathrm{Hor}(E)$ is a *horizontal* vector sub-bundle of $\mathsf{T}E$;

(∇2) for all $s \in \mathbb{R}$ and all $v_q \in E$, $\mathsf{T}\mu^s \cdot \mathrm{Hor}_{v_q}(E) = \mathrm{Hor}_{sv_q}(E)$, where $\mathrm{Hor}_{v_q}(E)$ denotes the fiber of $\mathrm{Hor}(E)$ over v_q and $\mu^s : E \to E$ is defined by $v_q \mapsto sv_q$.

We denote by $P_V : \mathsf{T}E \to \mathrm{Ver}(E)$ and $P_H : \mathsf{T}E \to \mathrm{Hor}(E)$ the projections induced by the Whitney sum decomposition in (∇1).

The condition (∇2) means that the horizontal sub-bundle $\mathrm{Hor}(E)$ is invariant by $\mathsf{T}\mu^s$, for all $s \in \mathbb{R}$. So is the vertical bundle $\mathrm{Ver}(E)$, since $\mu^s : E \to E$ preserves fibers. It then follows that $\mathsf{T}\mu^s$ commutes with the projections P_V and P_H.

As $\mathrm{Ver}(E) = \mathrm{Ker}\ \mathsf{T}\pi_E$, condition ($\nabla$1) implies that the restriction of the vector bundle epimorphism $\mathsf{T}\pi_E : \mathsf{T}E \to \mathsf{TM}$ to the horizontal sub-bundle $\mathrm{Hor}(E)$ is an epimorphism of smooth vector bundles whose restrictions to to the fibers are linear isomorphisms, i.e. $\mathsf{T}\pi_E : \mathrm{Hor}_{v_q} \to \mathsf{T}_q\mathsf{M}$ is a linear isomorphism for all $v_q \in E$. Given $v_q \in E$, we designate by $\mathrm{H}_{v_q} : \mathsf{T}_q\mathsf{M} \to \mathrm{Hor}_{v_q}(E)$ the inverse of $\mathsf{T}\pi_E : \mathrm{Hor}_{v_q}(E) \to \mathsf{T}_q\mathsf{M}$, called the *horizontal lift at* v_q. For all $q \in \mathsf{M}$, $s \in \mathbb{R}$, $v_q \in E_q$ and $z_q \in \mathsf{T}_q\mathsf{M}$, we have $\mathsf{T}\mu^s \cdot \mathrm{H}_{v_q}(z_q) = \mathrm{H}_{sv_q}(z_q)$.

A connection on $\pi_E : E \to \mathsf{M}$ defines a smooth vector bundle epimorphism:

$$\begin{tikzcd}
\mathsf{T}E \arrow[r, "\widetilde{\kappa_E}"] \arrow[d, "\tau_E"'] \arrow[dr, phantom, "\circlearrowleft"] & E \oplus_{\mathsf{M}} E \arrow[d, "\pi_1"] \\
E \arrow[r, "\mathrm{id}_E"'] & E
\end{tikzcd}$$

given by $\widetilde{\kappa_E} \doteq \widetilde{\kappa_E^V} \circ P_V$. We denote by κ_E the composite $\pi_2 \circ \widetilde{\kappa_E} : \mathsf{T}E \to E$; κ_E is called the *connector* of the connection $\mathrm{Hor}(E)$. The restriction of the connector to the vertical sub-bundle is independent of the connection, since it coincides with the inverse of the vertical lift. Note that, for all $v_q \in E$ and $X_{v_q} \in \mathsf{T}_{v_q}E$: $X_{v_q} = \mathrm{H}_{v_q}(\mathsf{T}\pi_E \cdot X_{v_q}) + \lambda_{v_q}(\kappa_E \cdot X_{v_q})$.

Given $z_q \in \mathsf{TM}$, the connection $\mathrm{Hor}(E)$ defines a map $\nabla^E_{z_q} : \Gamma^\infty(E) \to E_q$, by $\nabla^E_{z_q} X \doteq \kappa_E \cdot \mathsf{T}X \cdot z_q$. This map is a derivation, i.e. for all $f \in \mathfrak{F}(\mathsf{M})$ and all $X \in \Gamma^\infty(E)$, $\nabla^E_{z_q} fX = f(q)\nabla^E_{z_q} X + z_q[f]X(q)$. We can then define a map $\nabla^E : \mathfrak{X}(\mathsf{M}) \times \Gamma^\infty(E) \to \Gamma^\infty(E)$, by $\big(\nabla^E_X Y\big)(q) \doteq \nabla^E_{X(q)} Y \in E_q$, for all $q \in \mathsf{M}$. Then ∇^E is $\mathfrak{F}(\mathsf{M})$-linear on the first factor, and a derivation on the second. Reciprocally, given a map $\nabla^E : \mathfrak{X}(\mathsf{M}) \times \Gamma^\infty(E) \to \Gamma^\infty(E)$ which is $\mathfrak{F}(\mathsf{M})$-linear on the first factor and a derivation on the second, there exists a unique connection $\mathrm{Hor}(E)$ which induces ∇^E. The map ∇^E is also called a connection.

Given a curve $\gamma : I \to \mathsf{M}$ defined on the interval $I \subset \mathbb{R}$, we define the *covariant derivative* ∇^E_t along γ, induced by the connection, by $\nabla^E_t : X \in \Gamma^\infty(\gamma^* E) \mapsto \kappa_E \cdot \frac{TX}{dt} \in \Gamma^\infty(\gamma^* E)$, where $\gamma^* E$ denotes the *pull back* vector bundle.

Given a curve γ on M, $t_0 \in \mathrm{dom}\,\gamma$ and $v \in E_{\gamma(t_0)}$, there exists a unique section $X \in \Gamma^\infty(\gamma^* E)$ such that $X(t_0) = v$ and $\nabla^E_t X \equiv 0$; X is said to be obtained by *parallel translation* of v along γ, and we use the notation $\big(\forall\, t \in \mathrm{dom}\,\gamma\big)\; X(t) = \tau^\gamma_{t_0,t}(v)$. Besides, given $t_1 \in \mathrm{dom}\,\gamma$, the map $\tau^\gamma_{t_0,t_1} : E_{\gamma(t_0)} \to E_{\gamma(t_1)}$ defined by $v \mapsto \tau^\gamma_{t_0,t_1}(v)$ is a linear isomorphism. Using parallel translation, we can compute the horizontal lift at $v_q \in E$ by, for all $z_q \in \mathsf{T}_q\mathsf{M}$, $\mathrm{H}_{v_q}(z_q) = \frac{T}{dt}_{|t=0}\, \tau^\gamma_{0,t}(v_q)$, where $\gamma : (-\epsilon, \epsilon) \to \mathsf{M}$ is a curve on M with $\frac{T\gamma}{dt}_{|t=0} = z_q$.

A *connection on the smooth manifold* M is a connection ∇ on its tangent bundle $\tau_{\mathsf{M}} : \mathsf{TM} \to \mathsf{M}$. Such a connection defines a *spray* $\mathsf{S} \in \mathfrak{X}(\mathsf{TM})$, by $\mathsf{S}(v_q) = \mathrm{H}_{v_q}(v_q)$ - the so-called *geodesic spray* induced by the connection ∇. The *geodesics* of (M, ∇) are the *base integral curves* (i.e. the projections on M of its integral curves) of the second order vector field S, that is to say, the curves γ on M which satisfy $\nabla_t \frac{T\gamma}{dt} = 0$.

Given a connection $\mathrm{Hor}(E)$ on the smooth vector bundle $\pi_E : E \to \mathsf{M}$, the *curvature tensor* $\mathsf{R}^E : \mathfrak{X}(\mathsf{M}) \times \mathfrak{X}(\mathsf{M}) \times \Gamma^\infty(E) \to \Gamma^\infty(E)$, induced by the connection, is defined by $\mathsf{R}(X, Y) \cdot Z \doteq \nabla^E_X \nabla^E_Y \xi - \nabla^E_Y \nabla^E_X \xi - \nabla^E_{[X,Y]} \xi$, for all

$X, Y \in \mathfrak{X}(\mathsf{M}), \xi \in \Gamma^\infty(E)$. The connection is said to be *flat* if its curvature tensor R^E vanishes identically. For a connection ∇ on a smooth manifold M, we also define its *torsion tensor* $\mathsf{T} : \mathfrak{X}(\mathsf{M}) \times \mathfrak{X}(\mathsf{M}) \to \mathfrak{X}(\mathsf{M})$ by $\mathsf{T}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$. Such a connection is called *symmetric* or *torsionless* if its torsion tensor vanishes identically.

Finally, given $n \in \mathbb{N}$, there exists a flat connection naturally defined on the trivial vector bundle $\mathbb{R}^n_{\mathsf{M}}$: for each $(q, v) \in \mathbb{R}^n_{\mathsf{M}} = \mathsf{M} \times \mathbb{R}$, we have a canonical linear isomorphism $\mathsf{T}_{(q,v)}\mathbb{R}^n_{\mathsf{M}} \equiv \mathsf{T}_q\mathsf{M} \oplus \mathbb{R}^n$, and the second factor of this direct sum can be naturally identified with the vertical subspace at (q, v). We then define the horizontal subspace $\mathrm{Hor}_{(q,v)}(\mathbb{R}^n_{\mathsf{M}})$ as the first factor of this direct sum.

3 The Parallel Derivative

Let $\pi_E : E \to \mathsf{M}$ and $\pi_F : F \to \mathsf{N}$ be smooth vector bundles over M and N, respectively, and let $b : E \to F$ be a morphism of smooth fiber bundles (i.e. it preserves fibers and is smooth, but it needs not be linear on the fibers) over $\tilde{b} : \mathsf{M} \to \mathsf{N}$. We denote by $\mathbb{F}b : E \to \mathsf{L}(E, \tilde{b}^* F)$ the *fiber derivative* (see [1]) of b, that is to say, the morphism of smooth fiber bundles defined by, for all $w_q \in E_q$, $\mathbb{F}b(v_q) \cdot w_q \doteq \kappa^V_F \cdot \mathsf{T}b \cdot \lambda_{v_q}(w_q) \in F_{\tilde{b}(q)}$.

Let $\pi_E : E \to \mathsf{M}$ and $\pi_F : F \to \mathsf{N}$ be vector bundles endowed with connections $\mathrm{Hor}(E)$ and $\mathrm{Hor}(F)$, respectively. We now define the parallel derivative of b[1]:

Definition 1. *The differentiable fiber bundle morphism* $\mathbb{P}b : E \to \mathsf{L}(\mathsf{TM}, \tilde{b}^* F)$ *defined by, for all* $v_q \in E$ *and all* $z_q \in \mathsf{T}_q\mathsf{M}$*:*

$$\mathbb{P}b(v_q) \cdot z_q \doteq \kappa_F \cdot \mathsf{T}b \cdot \mathrm{H}_{v_q}(z_q) \in F_{\tilde{b}(q)}$$

is called the parallel derivative *of* b.

Roughly speaking, the usefulness of the fiber and parallel derivatives consists in providing a coordinate-free technique to compute the tangent map of b,

[1]this definition was suggested in [9], [8], [7].

allowing its computation at a given element of $\mathsf{T}E$ in terms of its vertical and horizontal components. That is to say, for all $X_{v_q} \in \mathsf{T}E$, the following formulae hold:

$$\begin{aligned} \mathsf{T}\pi_F \cdot \mathsf{T}b \cdot X_{v_q} &= \mathsf{T}\tilde{b} \cdot \mathsf{T}\pi_E \cdot X_{v_q} \\ \kappa_F \cdot \mathsf{T}b \cdot X_{v_q} &= \mathbb{F}b(v_q) \cdot \kappa_E \cdot X_{v_q} + \mathbb{P}b(v_q) \cdot \mathsf{T}\pi_E \cdot X_{v_q}. \end{aligned}$$

Therefore, given a curve γ on M and $X \in \Gamma^\infty(\gamma^* E)$, we have:

$$\nabla_t^F (b \circ X) = \mathbb{F}b(X) \cdot \nabla_t^E X + \mathbb{P}b(X) \cdot \frac{T\gamma}{dt}$$

Note that the connection ∇^F induces a connection $\nabla^{\tilde{b}^* F}$ in the pull back $\tilde{b}^* F$; for all $z_q \in \mathsf{TM}$ and $X \in \Gamma^\infty(\tilde{b}^* F)$, we define $\nabla_{z_q}^{\tilde{b}^* F} X \doteq \kappa_F \cdot \mathsf{T}X \cdot z_q \in F_{\tilde{b}(q)}$. Besides, if $\pi_G : G \to \mathsf{M}$ is another differentiable vector bundle over M, endowed with a connection ∇^G, we define a connection on the differentiable vector bundle $\mathsf{L}(E, G) \to \mathsf{M}$ in the following way: for all $A \in \Gamma^\infty\big(\mathsf{L}(E, G)\big)$, $Z \in \mathfrak{X}(\mathsf{M})$ and $X \in \Gamma^\infty(E)$, $\big(\nabla_Z^{\mathsf{L}(E,G)} A\big) \cdot X \doteq \nabla_Z^G \{A(X)\} - A\big(\nabla_Z^E X\big) \in \Gamma^\infty(G)$.

Let us now fix a connection ∇ on M. We then have connections defined on the smooth vector bundles $\mathsf{L}(E, \tilde{b}^* F)$ and $\mathsf{L}(\mathsf{TM}, \tilde{b}^* F)$, naturally induced by the connections on E and F and by the connection on M. Through the use of these connections, we can consider the fiber and parallel derivatives of the morphisms $\mathbb{F}b : E \to \mathsf{L}(E, \tilde{b}^* F)$ and $\mathbb{P}b : E \to \mathsf{L}(\mathsf{TM}, \tilde{b}^* F)$, obtaining the following differentiable fiber bundle morphisms:

$$\begin{aligned} \mathbb{FF}b &: E \to \mathsf{L}\big(E, \mathsf{L}(E, \tilde{b}^* F)\big) \equiv \mathsf{L}(E \otimes E, \tilde{b}^* F) \\ \mathbb{PF}b &: E \to \mathsf{L}\big(\mathsf{TM}, \mathsf{L}(E, \tilde{b}^* F)\big) \equiv \mathsf{L}(\mathsf{TM} \otimes E, \tilde{b}^* F) \\ \mathbb{FP}b &: E \to \mathsf{L}\big(E, \mathsf{L}(\mathsf{TM}, \tilde{b}^* F)\big) \equiv \mathsf{L}(E \otimes \mathsf{TM}, \tilde{b}^* F) \\ \mathbb{PP}b &: E \to \mathsf{L}\big(\mathsf{TM}, \mathsf{L}(\mathsf{TM}, \tilde{b}^* F)\big) \equiv \mathsf{L}(\mathsf{TM} \otimes \mathsf{TM}, \tilde{b}^* F) \end{aligned}$$

The relation of these morphisms to each other and to the curvature tensors of the chosen connections is given by the following theorem:

Theorem A. *Using the notation above, let* R^E *and* R^F *be the curvature tensors of the connections on* E *and* F*, respectively, and* T *the torsion tensor of the connection on* M*. Then the following formulae hold:*

(i) For all $q \in \mathsf{M}$*,* $v_q, w_q, z_q \in E_q$*:*

$$\mathbb{F}^2 b(v_q) \cdot (w_q, z_q) = \mathbb{F}^2 b(v_q) \cdot (z_q, w_q)$$

(ii) For all $q \in \mathsf{M}$*,* $v_q, w_q \in E_q$*,* $z_q \in \mathsf{T}_q\mathsf{M}$*:*

$$\mathbb{F}\mathbb{P} b(v_q) \cdot (w_q, z_q) = \mathbb{P}\mathbb{F} b(v_q) \cdot (z_q, w_q)$$

(iii) For all $q \in \mathsf{M}$*,* $v_q \in E_q$*,* $w_q, z_q \in \mathsf{T}_q\mathsf{M}$*:*

$$\begin{aligned}\mathbb{P}^2 b(v_q) \cdot (w_q, z_q) = \mathbb{P}^2 b(v_q) \cdot (z_q, w_q) + \mathbb{F} b(v_q) \cdot \mathsf{R}^E(z_q, w_q) \cdot v_q + \\ + \mathbb{P} b(v_q) \cdot \mathsf{T}(z_q, w_q) + \mathsf{R}^F(\mathsf{T}\tilde{b} \cdot w_q, \mathsf{T}\tilde{b} \cdot z_q) \cdot b(v_q)\end{aligned}$$

Proof. (i) Given $q \in \mathsf{M}$, $v_q, w_q, z_q \in E_q$, by definition we have $\mathbb{F} b(v_q) \cdot z_q = \frac{d}{dt}_{|t=0} b(v_q + t z_q)$ and $\mathbb{F}^2 b(v_q) \cdot (z_q, w_q) = \big(\mathbb{F}(\mathbb{F} b)(v_q) \cdot z_q\big) \cdot w_q$. That is:

$$\begin{aligned}\mathbb{F}^2 b(v_q) \cdot (z_q, w_q) &= \frac{d}{dt}_{|t=0} \big(\mathbb{F} b(v_q + t z_q)\big) \cdot w_q = \\ &= \frac{d}{dt}_{|t=0} \big(\mathbb{F} b(v_q + t z_q) \cdot w_q\big) = \\ &= \frac{d}{dt}_{|t=0} \frac{d}{ds}_{|s=0} b(v_q + t w_q + s z_q) = \\ &= \frac{d}{ds}_{|s=0} \frac{d}{dt}_{|t=0} b(v_q + t w_q + s z_q) = \\ &= \mathbb{F}^2 f(v_q) \cdot (w_q, z_q)\end{aligned}$$

(ii) Let $q \in \mathsf{M}$, $v_q, w_q \in E_q$ and $z_q \in \mathsf{T}_q\mathsf{M}$. By definition: (1) $\mathbb{P} b(v_q) \cdot z_q = \nabla^F_{s|s=0} b\big(V(s)\big)$, where V is the parallel transport in E of v_q along a curve $\gamma_{z_q} : (-\epsilon, \epsilon) \to \mathsf{M}$ tangent to z_q at 0 (i.e. $\frac{T}{ds}_{|s=0} \gamma_{z_q} = z_q$) and (2) $\mathbb{F}\mathbb{P} b(v_q) \cdot (w_q, z_q) = \big(\mathbb{F}(\mathbb{P} b)(v_q) \cdot w_q\big) \cdot z_q = \big(\frac{d}{dt}_{|t=0} \mathbb{P} b(v_q + t w_q)\big) \cdot z_q = \frac{d}{dt}_{|t=0} \big(\mathbb{P} b(v_q + t w_q) \cdot z_q\big) = \nabla^F_{t|t=0} \big(\mathbb{P} b(v_q + t w_q) \cdot z_q\big)$. In the last equality we have used that, given $q \in \mathsf{M}$ and a curve $t \mapsto X(t)$ on E_q, then the derivative $\frac{d}{dt} X$ coincides with the covariant derivative $\nabla^E_t X$, considering X as a section of E along the constant curve

$t \mapsto q$ on M. To compute $\mathbb{P}b(v_q + tw_q) \cdot z_q$, let $V : (-\tau, \tau) \times (-\epsilon, \epsilon) \to E$ such that $V(t, \cdot)$ is the parallel translation of $(v_q + tw_q)$ along γ_{z_q}, for each $t \in (-\tau, \tau)$. Then $\mathbb{P}b(v_q + tw_q) \cdot z_q = \nabla^F_{s|s=0} b\big(V(t,s)\big)$, so that:

$$\begin{aligned}
\mathbb{FP}b(v_q) \cdot (w_q, z_q) &= \nabla^F_{t|t=0} \nabla^F_{s|s=0} b\big(V(t,s)\big) = \\
&= \nabla^F_{s|s=0} \nabla^F_{t|t=0} b\big(V(t,s)\big) + \underbrace{\mathsf{R}^F(0, \mathsf{T}\tilde{b} \cdot z_q) \cdot b(v_q)}_{=0} = \\
&= \nabla^F_{s|s=0} \Big\{ \mathbb{F}b\big(V(0,s)\big) \cdot \nabla^E_{t|t=0} V(t,s) + \\
&\quad + \mathbb{P}b\big(V(0,s)\big) \cdot \underbrace{\frac{T}{dt}_{|t=0} \gamma_{z_q}(s)}_{=0} \Big\} = \\
&= \mathbb{PF}b(v_q) \cdot (z_q, w_q) + \mathbb{F}b(v_q) \cdot \underbrace{\nabla^E_{s|s=0} \nabla^E_{t|t=0} V(t,s)}_{= \mathsf{R}^E(z_q, 0) \cdot v_q = 0}
\end{aligned}$$

Hence, $\mathbb{FP}b(v_q) \cdot (w_q, z_q) = \mathbb{PF}b(v_q) \cdot (z_q, w_q)$, as asserted.

(iii) Let $q \in \mathsf{M}$, $v_q \in E_q$ and $w_q, z_q \in \mathsf{T}_q\mathsf{M}$. By definition, $\mathbb{P}^2 b(v_q) \cdot (w_q, z_q) = \big(\mathbb{P}(\mathbb{P}b)(v_q) \cdot w_q\big) \cdot z_q$. To compute $\mathbb{P}(\mathbb{P}b)(v_q) \cdot w_q$, let $\gamma_{w_q} : (-\tau, \tau) \to \mathsf{M}$ such that $\frac{T}{dt}_{|t=0} \gamma_{w_q} = w_q$ and $V(t)$ the parallel translation on E of v_q along γ_{w_q}. Then $\mathbb{P}(\mathbb{P}b)(v_q) \cdot w_q = \nabla^{\mathsf{L}(\mathsf{TM}, b^*F)}_{t|t=0} \mathbb{P}b(V(t))$, hence $\mathbb{P}^2 b(v_q) \cdot (w_q, z_q) = \big(\nabla^{\mathsf{L}(\mathsf{TM}, b^*F)}_{t|t=0} \mathbb{P}b(V(t))\big) \cdot z_q$. Let $Z(t)$ be the parallel translation on TM of z_q along γ_{w_q}. By the definition of the induced connection on $\mathsf{L}(\mathsf{TM}, b^*F)$, we have $\nabla^F_{t|t=0}\big(\mathbb{P}b(V(t)) \cdot Z(t)\big) = \big(\nabla^{\mathsf{L}(\mathsf{TM}, b^*F)}_{t|t=0} \mathbb{P}b(V(t))\big) \cdot z_q + \mathbb{P}b(v_q) \cdot \nabla^{\mathsf{TM}}_{t|t=0} Z(t) = \big(\nabla^{\mathsf{L}(\mathsf{TM}, b^*F)}_{t|t=0} \mathbb{P}b(V(t))\big) \cdot z_q$, since Z is parallel. On the other hand, to compute $\mathbb{P}b(V(t)) \cdot Z(t)$, let $\Gamma : (-\tau, \tau) \times (-\epsilon, \epsilon) \to \mathsf{M}$ be such that $\frac{T}{ds}_{|s=0} \Gamma(t,s) = Z(t)$ for each $t \in (-\tau, \tau)$, and $V : (-\tau, \tau) \times (-\epsilon, \epsilon) \to E$ such that $V(t, \cdot)$ is the parallel translation of $V(t)$ along $\Gamma(t, \cdot)$ for each $t \in (-\tau, \tau)$. Then $\mathbb{P}b(V(t)) \cdot Z(t) = \nabla^F_{s|s=0} b(V(t,s))$, so that:

$$\begin{aligned}
\mathbb{P}^2 b(v_q)\cdot(w_q,z_q) &= \nabla^F_{t|t=0}\big\{\mathbb{P}b\big(V(t)\big)\cdot Z(t)\big\} = \\
&= \nabla^F_{t|t=0}\big\{\nabla^F_{s|s=0} b\big(V(t,s)\big)\big\} = \\
&= \nabla^F_{s|s=0}\big\{\nabla^F_{t|t=0} b\big(V(t,s)\big)\big\} + \mathsf{R}^F(\mathsf{T}\tilde{b}\cdot w_q, \mathsf{T}\tilde{b}\cdot z_q)\cdot b(v_q) = \\
&= \nabla^F_{s|s=0}\big\{\mathbb{F}b\big(V(0,s)\big)\cdot\nabla^E_{t|t=0}V(t,s) + \\
&\quad + \mathbb{P}b\big(V(0,s)\big)\cdot\frac{T}{dt}_{|t=0}\Gamma(t,s)\big\} + \mathsf{R}^F(\mathsf{T}\tilde{b}\cdot w_q, \mathsf{T}\tilde{b}\cdot z_q)\cdot b(v_q) = \\
&= \mathbb{F}^2 b(v_q)\cdot(0,0) + \mathbb{P}\mathbb{F}b(v_q)\cdot(z_q,0) + \mathbb{F}b(v_q)\cdot\mathsf{R}^E(z_q,w_q)\cdot v_q + \\
&\quad + \mathbb{P}^2 b(v_q)\cdot(z_q,w_q) + \mathbb{P}b(v_q)\cdot\nabla^{\mathsf{TM}}_{s|s=0}\frac{T}{dt}_{|t=0}\Gamma(t,s) + \\
&\quad + \mathsf{R}^F(\mathsf{T}\tilde{b}\cdot w_q, \mathsf{T}\tilde{b}\cdot z_q)\cdot b(v_q)
\end{aligned}$$

But $\nabla^{\mathsf{TM}}_{s|s=0}\frac{T}{dt}_{|t=0}\Gamma(t,s) = \nabla^{\mathsf{TM}}_{t|t=0}\frac{T}{ds}_{|s=0}\Gamma(t,s) + \mathsf{T}(z_q,w_q) = \nabla^{\mathsf{TM}}_{t|t=0}Z + \mathsf{T}(z_q,w_q) = \mathsf{T}(z_q,w_q)$ (since Z is parallel). We have then obtained $\mathbb{P}^2 b(v_q)\cdot(w_q,z_q) = \mathbb{P}^2 b(v_q)\cdot(z_q,w_q) + \mathbb{F}b(v_q)\cdot\mathsf{R}^E(z_q,w_q)\cdot v_q + \mathbb{P}b(v_q)\cdot\mathsf{T}(z_q,w_q) + \mathsf{R}^F(\mathsf{T}\tilde{b}\cdot w_q, \mathsf{T}\tilde{b}\cdot z_q)\cdot b(v_q)$, as asserted.

4 Examples and Applications

In this section we present two examples of how the fiber and parallel derivatives can be used to make intrinsic computations. Firstly, we compute a formula for the Lie bracket of vector fields on the total space of a smooth vector bundle, and we use this formula to reobtain a well known characterization of flat connections. Secondly, we reobtain a well known formula for the canonical symplectic form in the tangent bundle of a Riemannian manifold in terms of vertical and horizontal components of tangent vectors. The reader is referred to [7], [9] and [8] for other examples and applications.

4.1 The Lie Bracket of Vector Fields on a Vector Bundle

As an application of theorem A, given a smooth vector bundle $\pi_E : E \to \mathsf{M}$ over M, endowed with a connection $\mathrm{Hor}(E)$, we obtain a formula for the Lie bracket of two vector fields $X, Y \in \mathfrak{X}(E)$ in terms of its horizontal and vertical components.

Firstly, given $f \in \mathfrak{F}(E)$, let us consider the differentiable fiber bundle morphism $\widetilde{f} : E \to \mathbb{R}_{\mathsf{M}}$ given by $\widetilde{f}(v_q) \doteq \big(q, f(q)\big)$, and let $\mathrm{Hor}(\mathbb{R}_{\mathsf{M}})$ be the trivial connection on $\mathbb{R}_{\mathsf{M}}$. Then, given $v_q \in E$ and $X_{v_q} \in \mathsf{T}_q E$, we have:

$$df(v_q) \cdot X_{v_q} = \kappa_{\mathbb{R}_{\mathsf{M}}} \cdot \mathsf{T}_{v_q}\widetilde{f} \cdot X_{v_q} = \mathbb{F}\widetilde{f}(v_q) \cdot \kappa_E \cdot X_{v_q} + \mathbb{P}\widetilde{f}(v_q) \cdot \mathsf{T}\pi_E \cdot X_{v_q}.$$

We shall omit henceforth the "$\sim$" in this notation, tacitly identifying f with $\widetilde{f}$, and we make use of this formula to calculate df.

Proposition 1. *Using the notation above, for all $X, Y \in \mathfrak{X}(E)$ and $v_q \in E$, we have:*

$$\begin{aligned}
\kappa_E \cdot [X,Y](v_q) &= \mathbb{F}(\kappa_E \circ Y)(v_q) \cdot \kappa_E \cdot X(v_q) + \mathbb{P}(\kappa_E \circ Y)(v_q) \cdot \mathsf{T}\pi_E \cdot X(v_q) - \\
&\quad - \mathbb{F}(\kappa_E \circ X)(v_q) \cdot \kappa_E \cdot Y(v_q) - \mathbb{P}(\kappa_E \circ X)(v_q) \cdot \mathsf{T}\pi_E \cdot Y(v_q) + \\
&\quad + \mathsf{R}^E\big(\mathsf{T}\pi_E \cdot Y(v_q), \mathsf{T}\pi_E \cdot X(v_q)\big) \cdot v_q \\
\mathsf{T}\pi_E \cdot [X,Y](v_q) &= \mathbb{F}(\mathsf{T}\pi_E \circ Y)(v_q) \cdot \kappa_E \cdot X(v_q) + \mathbb{P}(\mathsf{T}\pi_E \circ Y)(v_q) \cdot \mathsf{T}\pi_E \cdot X(v_q) - \\
&\quad - \mathbb{F}(\mathsf{T}\pi_E \circ X)(v_q) \cdot \kappa_E \cdot Y(v_q) - \mathbb{P}(\mathsf{T}\pi_E \circ X)(v_q) \cdot \mathsf{T}\pi_E \cdot Y(v_q)
\end{aligned}$$

Proof. We consider a torsionless connection on M to compute $\mathbb{P}^2$. Given $f \in \mathfrak{F}(E)$, $q \in \mathsf{M}$ and $v_q \in E_q$, we have $Y(v_q)[f] = df(v_q) \cdot Y(v_q) = \mathbb{F}f(v_q) \cdot \kappa_E \cdot Y(v_q) + \mathbb{P}f(v_q) \cdot \mathsf{T}\pi_E \cdot Y(v_q)$. Then, for all $v_q, w_q \in E_q$:

$$\begin{aligned}
\mathbb{F}(Y[f])(v_q) \cdot w_q &= \frac{d}{dt}_{|t=0} \{\mathbb{F}f(v_q + tw_q) \cdot \kappa_E \circ Y(v_q + tw_q) + \\
&\qquad + \mathbb{P}f(v_q + tw_q) \cdot \mathsf{T}\pi_E \circ Y(v_q + tw_q)\} = \\
&= \mathbb{F}^2 f(v_q) \cdot \big(w_q, \kappa_E \cdot Y(v_q)\big) + \mathbb{F}f(v_q) \cdot \mathbb{F}(\kappa_E \circ Y)(v_q) \cdot w_q + \\
&\quad + \mathbb{F}\mathbb{P}f(v_q) \cdot \big(w_q, \mathsf{T}\pi_E \cdot Y(v_q)\big) + \mathbb{P}f(v_q) \cdot \mathbb{F}(\mathsf{T}\pi_E \circ Y)(v_q) \cdot w_q
\end{aligned}$$

and, for all $v_q \in E_q, w_q \in \mathsf{T}_q\mathsf{M}$, taking $\gamma_{w_q} : (-\epsilon, \epsilon) \to \mathsf{M}$ such that $\frac{T\gamma_{w_q}}{dt}_{|t=0} = w_q$ and V parallel translation on E of v_q along γ_{w_q}:

$$\begin{aligned}
\mathbb{P}(Y[f])(v_q) \cdot w_q &= \frac{d}{dt}_{|t=0} \{\mathbb{F}f(V) \cdot \kappa_E \circ Y(V) + \mathbb{P}f(V) \cdot \mathsf{T}\pi_E \circ Y(V)\} = \\
&= \mathbb{P}\mathbb{F}f(v_q) \cdot \big(w_q, \kappa_E \cdot Y(v_q)\big) + \mathbb{F}f(v_q) \cdot \mathbb{P}(\kappa_E \circ Y)(v_q) \cdot w_q + \\
&\quad + \mathbb{P}^2 f(v_q) \cdot \big(w_q, \mathsf{T}\pi_E \cdot Y(v_q)\big) + \mathbb{P}f(v_q) \cdot \mathbb{P}(\mathsf{T}\pi_E \circ Y)(v_q) \cdot w_q
\end{aligned}$$

Hence, for all $v_q \in E$, we have:

$$
\begin{aligned}
[X,Y](v_q)[f] &= X(v_q)[Y[f]] - Y(v_q)[X[f]] = \\
&= \mathbb{F}(Y[f])(v_q)\cdot\kappa_E\cdot X(v_q) + \mathbb{P}(Y[f])(v_q)\cdot\mathsf{T}\pi_E\cdot X(v_q) - \\
&\quad - \mathbb{F}(X[f])(v_q)\cdot\kappa_E\cdot Y(v_q) - \mathbb{P}(X[f])(v_q)\cdot\mathsf{T}\pi_E\cdot Y(v_q) = \\
&= \mathbb{F}^2 f(v_q)\cdot\big(\kappa_E\cdot X(v_q),\kappa_E\cdot Y(v_q)\big) + \mathbb{F}f(v_q)\cdot\mathbb{F}(\kappa_E\circ Y)(v_q)\cdot\kappa_E\cdot X(v_q) + \\
&\quad + \mathbb{F}\mathbb{P}f(v_q)\cdot\big(\kappa_E\cdot X(v_q),\mathsf{T}\pi_E\cdot Y(v_q)\big) + \mathbb{P}f(v_q)\cdot\mathbb{F}(\mathsf{T}\pi_E\circ Y)(v_q)\cdot\kappa_E\cdot X(v_q) + \\
&\quad + \mathbb{P}\mathbb{F}f(v_q)\cdot\big(\mathsf{T}\pi_E\cdot X(v_q),\kappa_E\cdot Y(v_q)\big) + \mathbb{F}f(v_q)\cdot\mathbb{P}(\kappa_E\circ Y)(v_q)\cdot\mathsf{T}\pi_E\cdot X(v_q) + \\
&\quad + \mathbb{P}^2 f(v_q)\cdot\big(\mathsf{T}\pi_E\cdot X(v_q),\mathsf{T}\pi_E\cdot Y(v_q)\big) + \mathbb{P}f(v_q)\cdot\mathbb{P}(\mathsf{T}\pi_E\circ Y)(v_q)\cdot\mathsf{T}\pi_E\cdot X(v_q) - \\
&\quad - \big\{\mathbb{F}^2 f(v_q)\cdot\big(\kappa_E\cdot Y(v_q),\kappa_E\cdot X(v_q)\big) + \mathbb{F}f(v_q)\cdot\mathbb{F}(\kappa_E\circ X)(v_q)\cdot\kappa_E\cdot Y(v_q) + \\
&\quad + \mathbb{F}\mathbb{P}f(v_q)\cdot\big(\kappa_E\cdot Y(v_q),\mathsf{T}\pi_E\cdot X(v_q)\big) + \mathbb{P}f(v_q)\cdot\mathbb{F}(\mathsf{T}\pi_E\circ X)(v_q)\cdot\kappa_E\cdot Y(v_q) + \\
&\quad + \mathbb{P}\mathbb{F}f(v_q)\cdot\big(\mathsf{T}\pi_E\cdot Y(v_q),\kappa_E\cdot X(v_q)\big) + \mathbb{F}f(v_q)\cdot\mathbb{P}(\kappa_E\circ X)(v_q)\cdot\mathsf{T}\pi_E\cdot Y(v_q) + \\
&\quad + \mathbb{P}^2 f(v_q)\cdot\big(\mathsf{T}\pi_E\cdot Y(v_q),\mathsf{T}\pi_E\cdot X(v_q)\big) + \mathbb{P}f(v_q)\cdot\mathbb{P}(\mathsf{T}\pi_E\circ X)(v_q)\cdot\mathsf{T}\pi_E\cdot Y(v_q)\big\}
\end{aligned} \tag{1}
$$

On the other hand:

$$[X,Y](v_q)[f] = \mathbb{F}f(v_q)\cdot\kappa_E\cdot[X,Y](v_q) + \mathbb{P}f(v_q)\cdot\mathsf{T}\pi_E\cdot[X,Y](v_q) \tag{2}$$

Thus, by theorem A (note that $\mathrm{Hor}(\mathbb{R}_{\mathsf{M}})$ is flat) and by the arbitrariness of $f \in \mathfrak{F}(\mathsf{M})$, the proposition follows comparing equations (1) and (2).

As a corollary, we reobtain the following well known characterization of flat connections:

Corollary 1. *With the same notation, the horizontal sub-bundle* $\mathrm{Hor}(E)$ *is completely integrable if, and only if, the connection* $\mathrm{Hor}(E)$ *is flat.*

Proof. Indeed, given $X, Y \in \Gamma^\infty\big(\mathrm{Hor}(E)\big)$, we have $\kappa_E \circ X = \kappa_E \circ Y = 0$; consequently, it follows from proposition 1 that, $\big(\forall\, v_q \in E\big)\,\kappa_E\cdot[X,Y](v_q) = \mathsf{R}^E\big(\mathsf{T}\pi_E\cdot Y(v_q),\mathsf{T}\pi_E\cdot X(v_q)\big)\cdot v_q$. Hence, if the connection is flat, then $\kappa_E \circ [X,Y] \equiv 0$, i.e. $\mathrm{Hor}(E)$ is involutive. The same formula shows that, reciprocally, if $\mathrm{Hor}(E)$ is involutive, i.e. if $\kappa_E \circ [X,Y] \equiv 0$ for all $X, Y \in \Gamma^\infty\big(\mathrm{Hor}(E)\big)$, then

$\mathsf{R}^E \equiv 0$, by the arbitrariness $v_q \in E, X, Y \in \Gamma^\infty\big(\mathrm{Hor}(E)\big)$ and by the fact that $\mathsf{T}\pi_E|_{\mathrm{Hor}_{v_q}(E)} : \mathrm{Hor}_{v_q}(E) \to \mathsf{T}_q\mathsf{M}$ is a linear isomorphism.

4.2 The Symplectic Form on the Tangent Bundle of (M, g)

Let (M, g) be a Riemannian manifold. We denote by ω_0 the canonical symplectic form of the cotangent bundle of M, and by ω_{TM} its pull back by the Legendre transformation $\mathsf{g}^\flat : \mathsf{TM} \to \mathsf{T}^*\mathsf{M}$, $\mathsf{g}^\flat(v_q) \doteq \langle v_q, \cdot\rangle$. Let us consider on $\mathsf{T}^*\mathsf{M}$ the connection induced by the Levi-Civita connection of (M, g). With respect to these connections, the tangent map of the Legendre transformation, $\mathsf{Tg}^\flat : \mathsf{TTM} \to \mathsf{T}^*\mathsf{T}^*\mathsf{M}$, is a smooth vector bundle isomorphism which preserves the horizontal and vertical sub-bundles, i.e. $\mathsf{Tg}^\flat \cdot \mathrm{Hor}(\mathsf{TM}) = \mathrm{Hor}(\mathsf{T}^*\mathsf{M})$ and $\mathsf{Tg}^\flat \cdot \mathrm{Ver}(\mathsf{TM}) = \mathrm{Ver}(\mathsf{T}^*\mathsf{M})$. It then follows that $\mathsf{g}^\flat$ is natural with respect to the connectors, i.e. $\kappa_{\mathsf{T}^*\mathsf{M}} \circ \mathsf{Tg}^\flat = \mathsf{g}^\flat \circ \kappa_{\mathsf{TM}}$. We can then use the following corollary of proposition 1 to compute ω_0 and ω_{TM}:

Corollary 2. *Using the notation above, we have:*

(i) *For all $p_q \in \mathsf{T}^*\mathsf{M}$, $X_{p_q}, Y_{p_q} \in \mathsf{T}_{p_q}(\mathsf{T}^*\mathsf{M})$:*

$$\omega_0(X_{p_q}, Y_{p_q}) = \langle \mathsf{T}\tau_{\mathsf{T}^*\mathsf{M}} \cdot X_{p_q}, \kappa_{\mathsf{T}^*\mathsf{M}} \cdot Y_{p_q}\rangle - \langle \mathsf{T}\tau_{\mathsf{T}^*\mathsf{M}} \cdot Y_{p_q}, \kappa_{\mathsf{T}^*\mathsf{M}} \cdot X_{p_q}\rangle$$

(ii) *For all $v_q \in \mathsf{TM}$, $X_{v_q}, Y_{v_q} \in \mathsf{T}_{v_q}(\mathsf{TM})$:*

$$\omega_{\mathsf{TM}}(X_{v_q}, Y_{v_q}) = \langle \mathsf{T}\tau_{\mathsf{TM}} \cdot X_{v_q}, \kappa_{\mathsf{TM}} \cdot Y_{v_q}\rangle - \langle \mathsf{T}\tau_{\mathsf{TM}} \cdot Y_{v_q}, \kappa_{\mathsf{TM}} \cdot X_{v_q}\rangle$$

Proof. We need to show only part (i), since part (ii) is an immediate consequence of the formula from part (i) and of the following remarks: (1) $\omega_{\mathsf{TM}}(X_{v_q}, Y_{v_q}) = (\mathsf{g}^\flat)^*\omega_0(X_{v_q}, Y_{v_q}) = \omega_0(\mathsf{Tg}^\flat \cdot X_{v_q}, \mathsf{Tg}^\flat \cdot Y_{v_q})$, (2) $\kappa_{\mathsf{T}^*\mathsf{M}} \circ \mathsf{Tg}^\flat = \mathsf{g}^\flat \circ \kappa_{\mathsf{TM}}$ and $\mathsf{T}\tau_{\mathsf{T}^*\mathsf{M}} \circ \mathsf{Tg}^\flat = \mathsf{T}\tau_{\mathsf{TM}}$.

Let θ_0 be the canonical 1-form of the cotangent bundle of M, and $X, Y \in \mathfrak{X}(\mathsf{T}^*\mathsf{M})$ such that $X(p_q) = X_{p_q}, Y(p_q) = Y_{p_q}$. We have:

$$\begin{aligned} \omega_0(X_{p_q}, Y_{p_q}) &= -d\theta_0(X_{p_q}, Y_{p_q}) = \\ &= -X_{p_q}[\langle\theta_0, Y\rangle] + Y_{p_q}[\langle\theta_0, X\rangle] + \langle\theta_0(p_q), [X, Y](p_q)\rangle \end{aligned} \tag{3}$$

In order to calculate $X_{p_q}[\langle\theta_0, Y\rangle]$, note that $\big(\forall\, W_{p_q} \in \mathsf{T}_{p_q}(\mathsf{T}^*\mathsf{M})\big)$ $\langle\theta_0(p_q), W_{p_q}\rangle = \langle p_q, \mathsf{T}\tau_{\mathsf{T}^*\mathsf{M}} \cdot W_{p_q}\rangle$. A direct computation then shows that $\mathbb{F}\{\langle\theta_0, Y\rangle\}(p_q) \cdot \kappa_{\mathsf{T}^*\mathsf{M}} \cdot X_{p_q} = \langle\kappa_{\mathsf{T}^*\mathsf{M}} \cdot X_{p_q}, \mathsf{T}\tau_{\mathsf{T}^*\mathsf{M}} \cdot Y_{p_q}\rangle + \langle p_q, \mathbb{F}(\mathsf{T}\tau_{\mathsf{T}^*\mathsf{M}} \circ Y)(p_q) \cdot \kappa_{\mathsf{T}^*\mathsf{M}} \cdot X_{p_q}\rangle$ and $\mathbb{P}\{\langle\theta_0, Y\rangle\}(p_q) \cdot \mathsf{T}\tau_{\mathsf{T}^*\mathsf{M}} \cdot X_{p_q} = \langle p_q, \mathbb{P}(\mathsf{T}\tau_{\mathsf{T}^*\mathsf{M}} \circ Y)(p_q) \cdot \mathsf{T}\tau_{\mathsf{T}^*\mathsf{M}} \cdot X_{p_q}\rangle$. Thus:

$$\begin{aligned} X_{p_q}[\langle\theta_0, Y\rangle] &= \mathbb{F}\{\langle\theta_0, Y\rangle\}(p_q) \cdot \kappa_{\mathsf{T}^*\mathsf{M}} \cdot X_{p_q} + \mathbb{P}\{\langle\theta_0, Y\rangle\}(p_q) \cdot \mathsf{T}\tau_{\mathsf{T}^*\mathsf{M}} \cdot X_{p_q} = \\ &= \langle\kappa_{\mathsf{T}^*\mathsf{M}} \cdot X_{p_q}, \mathsf{T}\tau_{\mathsf{T}^*\mathsf{M}} \cdot Y_{p_q}\rangle + \langle p_q, \mathbb{F}(\mathsf{T}\tau_{\mathsf{T}^*\mathsf{M}} \circ Y)(p_q) \cdot \kappa_{\mathsf{T}^*\mathsf{M}} \cdot X_{p_q}\rangle + \\ &\quad + \langle p_q, \mathbb{P}(\mathsf{T}\tau_{\mathsf{T}^*\mathsf{M}} \circ Y)(p_q) \cdot \mathsf{T}\tau_{\mathsf{T}^*\mathsf{M}} \cdot X_{p_q}\rangle \end{aligned} \tag{4}$$

On the other hand, it follows from proposition 1 that:

$$\begin{aligned} \langle\theta_0(p_q), [X, Y](p_q)\rangle = \langle p_q, &\mathbb{F}(\mathsf{T}\tau_{\mathsf{T}^*\mathsf{M}} \circ Y)(p_q) \cdot \kappa_{\mathsf{T}^*\mathsf{M}} \cdot X_{p_q} + \\ &+ \mathbb{P}(\mathsf{T}\tau_{\mathsf{T}^*\mathsf{M}} \circ Y)(p_q) \cdot \mathsf{T}\tau_{\mathsf{T}^*\mathsf{M}} \cdot X_{p_q}\rangle - \\ - \langle p_q, &\mathbb{F}(\mathsf{T}\tau_{\mathsf{T}^*\mathsf{M}} \circ X)(p_q) \cdot \kappa_{\mathsf{T}^*\mathsf{M}} \cdot Y_{p_q} + \\ &+ \mathbb{P}(\mathsf{T}\tau_{\mathsf{T}^*\mathsf{M}} \circ X)(p_q) \cdot \mathsf{T}\tau_{\mathsf{T}^*\mathsf{M}} \cdot Y_{p_q}\rangle \end{aligned} \tag{5}$$

Substituting equations (4) and (5) in (3), the thesis follows.

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References

[1] Abraham, R. C., Marsden, J. E., Ratiu, T., *Manifolds, Tensor Analysis, and Applications*, no. 75 in Applied Mathematical Sciences, Springer-Verlag, New York, second ed., 1988.

[2] do Carmo, M. P., , *Riemannian Geometry*, Mathematics: Theory and Applications, Birkhauser, 1994.

[3] Helgason, S., , *Differential Geometry, Lie Groups, and Symmetric Spaces*, vol. 80 of Pure and Applied Mathematics, Academics Press, San Diego, 1978.

[4] Kobayashi, S., Nomizu, K., *Foundations of Differential Geometry*, Wiley Classics Library, Wiley-Interscience, New York, 1996.

[5] Lang, S., *Differential and Riemannian Manifolds*, vol. 160 of Graduate Texts in Mathematics, Springer Verlag, New York, third ed., 1995.

[6] Spivak, M., *A Comprehensive Introduction to Differential Geometry*, Publish or Perish, 1999.

[7] Terra, G., *Sistemas Mecânicos e Lagrangeanos com Vínculos não-Lineares*, PhD Thesis, Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, Brasil, 2003.

[8] Terra, T., Kobayashi, M. H.,, *On classical mechanical systems with non-linear constraints*, Journal of Geometry and Physics, 49 (2004), 385–417.

[9] ——, *On the variational mechanics with non-linear constraints*, Journal de Mathématiques Pures et Appliquées, 83 (2004), pp. 629–671.

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