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A NOTE ON THE FIRST EIGENVALUE OF SPHERICALLY SYMMETRIC MANIFOLDS

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Abstract

We give lower and upper bounds for the first eigenvalue of geodesic balls in spherically symmetric manifolds. These lower and upper bounds are C^0-dependent on the metric coefficients. It gives better lower bounds for the first eigenvalue of spherical caps than those from Betz-Camera-Gzyl.

1 Introduction

Let $B_{\mathbb{N}^n(\kappa)}(r)$ be a geodesic ball of radius $r > 0$ in the simply connected n-dimensional space form $\mathbb{N}^n(\kappa)$ of constant sectional curvature κ and let $\lambda_1(B_{\mathbb{N}^n(\kappa)}(r))$ be its first Laplacian eigenvalue, i.e. the smallest real number $\lambda = \lambda_1(B_{\mathbb{N}^n(\kappa)}(r))$ for which there exists a function, called a first eigenfunction, $u \in C^2(B_{\mathbb{N}^n(\kappa)}(r)) \cap C^0(\overline{B_{\mathbb{N}^n(\kappa)}(r)}) \setminus \{0\}$, satisfying $\triangle u + \lambda u = 0$ in $B_{\mathbb{N}^n(\kappa)}(r)$ with $u|\partial B_{\mathbb{N}^n(\kappa)}(r) = 0$. In the case $\kappa = 0$, it is well known that $\lambda_1(B_{\mathbb{R}^n}(r)) = (c(n)/r)^2$, where $c(n)$ is the first zero of the Bessel function $J_{n/2-1}$. In the case $\kappa = -1$, there are fairly good lower and upper bounds for $\lambda_1(B_{\mathbb{H}^n}(r))$. For instance, one has that

$$\begin{aligned}\sqrt{\lambda_1(B_{\mathbb{H}^n}(r))} \;\; &\leq \;\; (n-1)(\coth(r/2)-1)/2 + [(n-1)^2/4 + \\ &\quad +4\pi^2/r^2 + (n-1)^2(\coth(r/2)-1)^2/4]^{1/2},\end{aligned}$$

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see [6] page 49. For sharper upper bounds for $\lambda_1(B_{\mathbb{H}^n}(r))$, see [9]. On the other hand, one has the well known lower bound,

$$\sqrt{\lambda_1(B_{\mathbb{H}^n}(r))} \geq \frac{(n-1)\coth(r)}{2},$$

proved by McKean, [10], [13]. This lower bound was improved by Bessa and Montenegro in [4] to

$$\sqrt{\lambda_1(B_{\mathbb{H}^n}(r))} \geq \max\left\{\frac{n}{2r}, \frac{(n-1)\coth(r)}{2}\right\}. \tag{1}$$

The case $c = 1$ is more delicate. Although the sphere is a very well studied manifold, the values of the first Laplacian eigenvalue $\lambda_1(B_{\mathbb{S}^n}(r))$, (Dirichlet boundary data if $r < \pi$) are pretty much unknown, with the exceptions $\lambda_1(B_{\mathbb{S}^n}(\pi/2)) = n$, $\lambda_1(B_{\mathbb{S}^n}(\pi)) = 0$. In dimension two and three there are good lower bounds due to Barbosa-DoCarmo [1], Pinsky [11], Sato [12] and Friedland-Hayman [8]. In higher dimension, the only lower bounds known (to the best of our knowledge) are the following lower bounds due to Betz, Camera and Gzyl obtained in [5] via probabilistic methods.

$$\left(\frac{c(n)}{r}\right)^2 > \lambda_1(B_{\mathbb{S}^n}(r)) \geq \frac{1}{\displaystyle\int_0^r \left[\frac{1}{\sin^{n-1}(\sigma)} \cdot \int_0^\sigma \sin^{n-1}(s)ds\right] d\sigma}. \tag{2}$$

The upper bound is due to Cheng's eigenvalue comparison theorem [7] since the Ricci curvature of the sphere is positive (in fact, it needed only to be non-negative).

In order to state our result, recall the definition of a spherically symmetric manifold. Let M be a Riemannian manifold and a point $p \in M$. For each vector $\xi \in T_pM$, let γ_ξ be the unique geodesic satisfying $\gamma_\xi(0) = p$, $\gamma'_\xi(0) = \xi$ and $d(\xi) = \sup\{t > 0 : \mathrm{dist}_M(p, \gamma_\xi(t)) = t\}$. Let $\mathcal{D}_p = \{t\,\xi \in T_pM : 0 \leq t < d(\xi),\, |\xi| = 1\}$ be the largest open subset of T_pM such that for any $\xi \in \mathcal{D}_p$ the geodesic $\gamma_\xi(t) = \exp_p(t\,\xi)$ minimizes the distance from p to $\gamma_\xi(t)$ for all $t \in [0, d(\xi)]$. The cut locus of p is the set $\mathrm{Cut}(p) = \{\exp_p(d(\xi)\,\xi),\, \xi \in T_pM,\, |\xi| = 1\}$ and $M = \exp_p(\mathcal{D}_p) \cup \mathrm{Cut}(p)$.

The exponential map $\exp_p : \mathcal{D}_p \to \exp_p(\mathcal{D}_p)$ is a diffeomorphism and is called the geodesic coordinates of $M \backslash \mathrm{Cut}(p)$. Fix a vector $\xi \in T_pM$, $|\xi| = 1$ and denote by $\xi^\perp$ the orthogonal complement of $\{\mathbb{R}\xi\}$ in T_pM and let $\tau_t : T_pM \to T_{\exp_p(t\,\xi)}M$ be the parallel translation along γ_ξ. Define the path of linear transformations

$$\mathcal{A}(t,\xi) : \xi^\perp \to \xi^\perp$$

by

$$\mathcal{A}(t,\xi)\eta = (\tau_t)^{-1}Y(t)$$

where $Y(t)$ is the Jacobi field along γ_ξ determined by the initial data $Y(0) = 0$, $(\nabla_{\gamma'_\xi}Y)(0) = \eta$. Define the map

$$\mathcal{R}(t) : \xi^\perp \to \xi^\perp$$

by

$$\mathcal{R}(t)\eta = (\tau_t)^{-1}\,\mathrm{R}(\gamma'_\xi(t), \tau_t\ \eta)\gamma'_\xi(t),$$

where R is the Riemann curvature tensor of M. It turns out that the map $\mathcal{R}(t)$ is a self adjoint map and the path of linear transformations $\mathcal{A}(t,\xi)$ satisfies the Jacobi equation $\mathcal{A}'' + \mathcal{R}\mathcal{A} = 0$ with initial conditions $\mathcal{A}(0,\xi) = 0$, $\mathcal{A}'(0,\xi) = I$. On the set $\exp_p(\mathcal{D}_p)$ the Riemannian metric of M can be expressed by

$$ds^2(\exp_p(t\,\xi)) = dt^2 + |\mathcal{A}(t,\xi)d\xi|^2. \tag{3}$$

Definition 1.1 *A manifold M is said to be spherically symmetric if the matrix $\mathcal{A}(t,\xi) = f(t)I$, for a function $f \in C^2([0,R])$, $R \in (0,\infty]$ with $f(0) = 0$, $f'(0) = 1$, $f|(0,R) > 0$.*

The class of spherically symmetric manifolds includes the canonical space forms $\mathbb{R}^n$, $\mathbb{S}^n(1)$ and $\mathbb{H}^n(-1)$. The n-volume $V(r)$ of a geodesic ball $B_M(r)$ of radius r in a spherically symmetric manifold is given by $V(r) = w_n \int_0^r f^{n-1}(s)ds$, whereas the $(n-1)$-volume $S(r)$ of the boundary $\partial B_M(r)$ is given by $S(r) = w_n f^{n-1}(r)$. Here w_n denotes the $(n-1)$-volume of the sphere $\mathbb{S}^{n-1}(1) \subset \mathbb{R}^n$. The authors [2] obtained using fixed point methods the following lower bound for the first eigenvalue $\lambda_1(B_M(r))$ of geodesic balls $B_M(r)$ with radius r in a spherically symmetric manifold M,

$$\lambda_1(B_M(r)) \geq \frac{1}{\displaystyle\int_0^r \frac{V(\sigma)}{S(\sigma)}\,d\sigma}. \tag{4}$$

It is worth mentioning that this lower bound (4) is Betz-Camera-Gzyl's lower bound when $M = \mathbb{S}^n$. The purpose of this note is give upper and better lower bounds for $\lambda_1(B_M(r))$. We prove the following theorem.

Theorem 1.2 *Let $B_M(r) \subset M$ be a ball in a spherically symmetric Riemannian manifold with metric $dt^2 + f^2(t)d\theta^2$, where $f \in C^2([0,R])$ with $f(0) = 0$, $f'(0) = 1$, $f(t) > 0$ for all $t \in (0,R]$. For every non-negative function $u \in C^0([0,r])$ set*

$$h(t,u) = \left[u(t) \Big/ \int_t^r \int_0^\sigma \left(\frac{f(s)}{f(\sigma)}\right)^{n-1} u(s)\,ds\,d\sigma \right].$$

Then

$$\sup_t h(t,u) \geq \lambda_1(B_M(r)) \geq \inf_t h(t,u) \tag{5}$$

Equality holds in (5) if and only if u is a first positive eigenfunction of $B_M(r)$ and $\lambda_1(B_M(r)) = h(t,u)$.

We should remark that taking $u \equiv 1$ in (5) we obtain (4). In the following table we compare our estimates for $\lambda_1(r) = \lambda_1(B_{\mathbb{S}^n}(r))$ for $n = 2, 3$, $r = \pi/8, \pi/4, 3\pi/8, \pi/2, 5\pi/8$ taking $u(t) = \cos(t\pi/2r)$ with the estimates obtained by Betz-Camera-Gzyl.

$n=2$ | $r=$	$\pi/8$	$\pi/4$	$\pi/8$	$\pi/2$	$5\pi/8$
BCG | $\lambda_1(r)$	≥ 25.77	≥ 6.31	≥ 2.70	≥ 1.44	≥ 0.85
BB | $\lambda_1(r)$	≥ 35.85	≥ 8.78	≥ 3.76	$= 2$	≥ 1.01
$n=3$ | $r=$	$\pi/8$	$\pi/4$	$3\pi/8$	$\pi/2$	$5\pi/8$
BCG | $\lambda_1(r)$	≥ 38.50	≥ 9.31	≥ 3.90	≥ 2	≥ 1.10
BB | $\lambda_1(r)$	≥ 57.94	≥ 14.01	≥ 5.86	$= 3$	≥ 1.27

2 Proof of Theorem 1.2

We start recalling the following theorem due to J. Barta.

Theorem 2.1 (Barta, [3]) *Let $\Omega \subset M$ be a bounded domain with piecewise smooth boundary $\partial\,\Omega$ in a Riemannian manifold. For any $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with $f|\Omega > 0$ and $f|\partial\Omega = 0$ one has that*

$$\sup_M(-\triangle f/f) \geq \lambda_1(\Omega) \geq \inf_\Omega(-\triangle f/f). \tag{6}$$

Equality in (6) holds if and only if f is a first eigenfunction of Ω. The lower bound inequality needs only that $f|\Omega > 0$.

Let $u \in C^0([0,r])$, $u \geq 0$. Define a function $T(u) \in C^1([0,r])$ by $T(u)(t) = \int_t^r \int_0^\sigma (f(s)/f(\sigma))^{n-1} u(s) ds d\sigma$. Extend u and Tu radially to $B_M(r)$ by $\tilde{u}(\exp_p(t\,\eta)) = u(t)$ and $\tilde{T}(u)(\exp_p(t\,\eta)) = T(u)(t)$, for $\eta \in \mathbb{S}^{n-1}$. Observe that $\tilde{T}(u)(\exp_p(t\,\eta)) \geq 0$, with $\tilde{T}(u)(\exp_p(t\,\eta)) = 0$ if and only if $t = r$. We claim that

$$\triangle \tilde{T}u(\exp_p(t\,\eta)) = -\tilde{u}(\exp_p(t\,\eta)) \tag{7}$$

as the following straight forward computation shows.

Proof: The expression of a spherically symmetric metric in geodesic coordinates is given by $ds^2 = dt^2 + f^2(t)d\theta^2$. The Laplacian in these coordinates is given by

$$\triangle = \frac{\partial^2}{\partial t^2} + (n-1) \cdot \frac{f'(t)}{f(t)} \cdot \frac{\partial}{\partial t} + \frac{1}{f^2(t)} \triangle_{\mathbb{S}^{n-1}}$$

Observe that a geodesic ball $B_M(r)$ are covered by one geodesic chart. Since $\tilde{T}u(\exp_p(t\,\eta)) = T(u)(t)$, we have that

$$\frac{\partial^2}{\partial t^2} T(u)(t) = -u(t) + (n-1)\frac{f'(t)}{f^n(t)} \int_0^t f^{(n-1)}(s)u(s)ds$$

and

$$\frac{\partial}{\partial t} T(u)(t) = -\frac{1}{f^{(n-1)}(t)} \int_0^t f^{(n-1)}(s)u(s)ds.$$

Therefore we have that

$$\triangle \tilde{T}u(\exp_p(t\,\eta)) = -u(t) = \tilde{u}(\exp_p(t\,\eta)).$$

Applying Barta's Theorem we obtain that

$$\sup_t \frac{u}{T(u)}(t) \geq \lambda_1(B(r)) \geq \inf_t \frac{u}{T(u)}(t).$$

Barta's Theorem says that equality in the above inequality holds if and only if $\tilde{T}(u)$ is a first eigenfunction. Thus we need only to show that $\tilde{T}(u)$ is a first eigenfunction if and only if u is a first eigenfunction. Suppose that we have equality in (7) then $\tilde{T}(u)$ is an eigenfunction, this is

$$0 = \triangle\tilde{T}u + \lambda_1(B_M(r))\tilde{T}u = -\tilde{u} + \lambda_1(B_M(r))\tilde{T}u \tag{8}$$

Applying the Laplacian in both side of the equation (8) we obtain by equation (7) that

$$0 = -\triangle\tilde{u} + \lambda_1(B_M(r))\triangle\tilde{T}u = -(\triangle\tilde{u} + \lambda_1(B_M(r))\tilde{u}) \tag{9}$$

Therefore u is a first eigenfunction with $\lambda_1(B_M(r)) = \dfrac{u}{T(u)}\cdot$

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