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GENERALIZED WEINGARTEN SURFACES OF BRYANT TYPE IN HYPERBOLIC 3-SPACE

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Abstract

We introduce a large class of generalized Weingarten surfaces which includes the linear Weingarten surfaces of Bryant type in the hyperbolic 3-space. We present a method for obtaining such surfaces in terms of holomorphic data, which is based on seeing a surface in hyperbolic 3-space, as the envelope of a sphere congruence whose other envelope is contained in the ideal boundary. In particular we obtain a new method for constructing linear Weingarten surfaces of Bryant type. As an applications of the method, we construct four different one-parameter families of complete generalized Weingarten surfaces.

Introduction.

During the last two decades there has been a renewed interest in the geometry of surfaces in hyperbolic manifolds. Basic problems such as existence, symmetry, behaviour at infinity, global structure and classification of complete surfaces of constant curvature in the hyperbolic 3-space $\mathbb{H}^3$ have called the attention of several authors. Concerning constant mean curvature one surfaces in $\mathbb{H}^3$, also called *Bryant surfaces*. The paper of R. Bryant [Br], was crucial. He showed how to parametrize these surfaces by meromorphic data and he started studing of their geometry. Since Bryant's work many properties and examples have been discovered in [GG], [LR], [Sm], [ST2], [UY1] among others. In [GMM1] and [Ro], different ways of constructing flat surfaces in $\mathbb{H}^3$ are presented. In [GMM2], the authors extend Bryant's results to linear Weingarten surfaces in

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$\mathbb{H}^3$, whose mean curvature H and Gaussian curvature K_I satisfy a linear relation of the form $2a(H-1)+bK_I=0$ for some $a,b\in\mathbb{R}$, $a+b\neq 0$. These surfaces are called *linear Weingarten surfaces of Bryant type*, in short, BLW-surfaces and they are a particular case of the special surfaces studied by Sa Earp and Toubiana in [ST1].

In this paper we introduce a large class of generalized Weingarten surfaces in the hyperbolic 3-space which includes the linear Weingarten surfaces of Bryant type. This idea was motivated by Schief in [Sc] who studied two classes of surfaces in the Euclidean 3-space, whose mean curvature H and Gauss curvature K satisfy $1+\mu H+(\mu\pm\rho)K=0$, where ρ and μ are harmonic functions in a particular sense. Those surfaces are named *generalized Weingarten surfaces* and some of them are related to the description of gravitational and neutrino fields in general relativity. We consider $\mathbb{H}^3=\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_3>0\}$ as the model of the *upper half-space*, with the metric $ds^2=\frac{1}{x_3^2}(dx_1^2+dx_2^2+dx_3^2)$ and *ideal boundary* $\mathbb{C}_\infty=\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_3=0\}\cup\{\infty\}$. Now, let M be a connected and orientable Riemann surface, $X:M\to\mathbb{H}^3$ an immersion with associated Gauss map η and *hyperbolic Gauss map* G. That is, for any $p\in M$, the oriented geodesic emanating from $X(p)$ meets the ideal boundary $\mathbb{C}_\infty$ at $G(X(p))$. The hyperbolic Gauss map defines a sphere congruence, for which $X(M)$ and $G(X(M))$ are envelopes and the radius of each sphere defines the *radius function* h. Then $X:M\to\mathbb{H}^3$ is called a *generalized Weingarten surface of Bryant type* (in short, a BGW-surface) if the mean curvature H, the Gauss curvature K_I and radius function h satisfy the relation

$$2ach^{\frac{2(c-1)}{c}}(H-1)+(a+b-ach^{\frac{2(c-1)}{c}})K_I=0$$

for some $a,b,c\in\mathbb{R}, a+b\neq 0, c\neq 0$. We present a method for obtaining such surfaces, which is based on seeing a surface in $\mathbb{H}^3$, as the envelope of a sphere congruence whose other envelope is contained in $\mathbb{C}_\infty$; in this case a BGW-surface is determined by the radius function which is a solution of the Liouville equation. When $c=1$, the surfaces are BLW-surfaces and in this case the method for constructing the surfaces turns out to be different from the one used in [GMM2]. When $c = 1$ and $b = 0$ the surfaces are Bryant surfaces, and in this case the method here is different from that used in [Br], [GG], [LR], [Sm], [ST2] and [UY1]. When $c = 1$ and $a = 0$ the surfaces are flat surfaces and again the method of construction here is different from that used in [GMM2] and [Ro]. Further consequences of the ideas in this paper are explored in [Co2] and [Co3].

The paper is organized as follows: In section 1, we characterize locally the surfaces in $\mathbb{R}^3$ which are the envelope of a sphere congruence, whose other envelope is contained in a plane. These surfaces can be considered as Ribaucour transformations of a plane (see [CT]). The parametrization obtained is related to hydrodynamic type systems, studied in [DN].

In section 2, we introduce the BGW-surfaces in the hyperbolic 3-space which include the linear Weingarten surfaces of Bryant type. We show that these surfaces can be parametrized in terms of two holomorphic functions and their derivatives, requiring no integration. As an example we present two one-parameter families of complete flat surfaces. We also study the asymptotic behavior of a BGW-surface in $\mathbb{C}_\infty$.

In section 3, we present four one-parameter families of complete BGW-surfaces. In particular, we obtain two one-parameter families of complete linear Weingarten surfaces of Bryant type.

1. Sphere Congruence in $\mathbb{R}^3$.

In this section, we characterize a the surface which is envelopes of a sphere congruence, whose other envelope is contained in a plane.

Let $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a surface and $N : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ the Gauss map. Then

$$N_{,i} = \sum_{j=1}^{2} W_{ij}(u) X_{,j}, \quad 1 \le i \le 2,$$

where $u = (u_1, u_2) \in U \subset \mathbb{R}^2$, $X_{,j}$ denotes the partial derivative of X with respect to u_i and $W = (W_{ij})$ is called *Weingarten matrix* .

A *sphere congruence* in $\mathbb{R}^3$ is a two-parameter family of spheres with a differen-

tiable *radius function* , whose centers lie on a surface. An *envelope* of a sphere congruence is a surface M of $\mathbb{R}^3$ such that each point of M is tangent to a sphere of the sphere congruence.
Let $\Pi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\}$, $Y : U \subset \mathbb{R}^2 \to \Pi$ be an orthogonal local parametrization of Π, i. e., $L_{ij} =< Y_{,i}, Y_{,j} >$, $1 \le i, j \le 2$, is such that

$$L_{12} = 0, \quad L_{ii} \neq 0, \quad 1 \le i \le 2.$$

Furthermore

$$Y_{,ij} = \sum_{k=1}^{2} \Gamma_{ij}^{k} Y_{,k}, \quad 1 \le i, j \le 2. \tag{1}$$

Using the fact that the metric is diagonal, we have

$$\Gamma_{ij}^{i} = \frac{L_{ii,j}}{2L_{ii}}, \quad \text{for all} \quad i, j \qquad \Gamma_{ii}^{j} = -\frac{L_{ii}}{L_{jj}}\Gamma_{ij}^{i}, \quad 1 \le i \neq j \le 2. \tag{2}$$

According to [DN], the hydrodynamic type system associated to this zero-curvature metric, is

$$u_{,t}^{i} = \sum_{j=1}^{2} V_{ij}(u) u_{,x}^{j}, \quad 1 \le i \le 2,$$

where

$$V_{ij} = \frac{1}{L_{jj}} \left(h_{,ij} - \sum_{\ell=1}^{2} \Gamma_{ij}^{\ell} h_{,\ell} \right), \quad 1 \le i, j \le 2, \tag{3}$$

and $h : U \subset \mathbb{R}^2 \to \mathbb{R}$ is an arbitrary $C^\infty(U)$ function. The following result relates this system with a sphere congruence.

Theorem 1.1: *A surface M in $\mathbb{R}^3$ is the envelope of a sphere congruence, whose other envelope is contained in the plane $\Pi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\}$ if, and only if, there exist an orthogonal local parametrization of Π, $Y : U \subset \mathbb{R}^2 \to \Pi$ and a differentiable function $h : U \subset \mathbb{R}^2 \to \mathbb{R}$, such that $X : U \subset \mathbb{R}^2 \to M$, given by*

$$X(u) = Y(u) - \frac{2h(u)}{S} \left(\frac{h_{,1}}{L_{11}}(u) Y_{,1}(u) + \frac{h_{,2}}{L_{22}}(u) Y_{,2}(u) - e_3 \right), \tag{4}$$

is a parametrization of M, *with* $e_3 = (0,0,1)$, $L_{ii} =< Y_{,i}, Y_{,i} >, \quad 1 \leq i \leq 2$ *and*

$$S = \frac{(h_{,1}(u))^2}{L_{11}} + \frac{(h_{,2}(u))^2}{L_{22}} + 1. \tag{5}$$

moreover, the Gauss map is given by

$$N(u) = e_3 + \frac{2}{S}\left(\frac{h_{,1}}{L_{11}}(u)Y_{,1}(u) + \frac{h_{,2}}{L_{22}}(u)Y_{,2}(u) - e_3\right), \tag{6}$$

the Weingarten matrix is

$$W = \frac{2}{P}(SV - 2h\ \det(V)I), \tag{7}$$

where the matrix $V = (V_{ij})$ *is given by* (3) *and*

$$P = S^2 - 2hS\,\mathrm{tr}(V) + 4h^2\det(V) \neq 0. \tag{8}$$

Proof. A surface M in $\mathbb{R}^3$ is the envelope of a sphere congruence, whose other envelope is contained in the plane Π if, and only if, there exist a orthogonal local parametrization of Π, $Y : U \subset \mathbb{R}^2 \to \Pi$ and a differentiable function $h : U \subset \mathbb{R}^2 \to \mathbb{R}$, such that $X : U \subset \mathbb{R}^2 \to M$, is a parametrization of M satisfying

$$X(u) = Y(u) + h(u)(e_3 - N(u)), \tag{9}$$

where N is the Gauss map of X, such that

$$N = \sum_{j=1}^{2} B_j Y_{,j} + B_3 e_3, \tag{10}$$

and

$$\sum_{j=1}^{2}(B_j)^2 L_{jj} + (B_3)^2 = 1. \tag{11}$$

From (9), we obtain

$$X_{,i} = Y_{,i} + h_{,i}(e_3 - N) - hN_{,i}, \quad 1 \leq i \leq 2. \tag{12}$$

Hence, using (10) and $< N, X_{,i} >= 0$, we get

$$B_i = \frac{h_{,i}}{L_{ii}}(1 - B_3), \quad 1 \leq i \leq 2. \tag{13}$$

Using (11), we obtain

$$B_3 = 1 - \frac{2}{S},$$

where S is given by (5). Using this last equality, (13), (9) and (10) we obtain the expressions (4) and (6). In order to determine the Weingarten matrix of X, let us consider the following vector valued functions

$$C = Y + he_3, \quad D = \frac{h_{,1}}{L_{11}}(u)Y_{,1}(u) + \frac{h_{,2}}{L_{22}}(u)Y_{,2}(u) + (\frac{S}{2} - 1)e_3. \tag{14}$$

Using (3) and the expression of S given by (5), we have

$$D_{,i} = \sum_{k=1}^{2} V_{ik}C_{,k}, \quad S_{,i} = 2\sum_{k=1}^{2} V_{ik}h_{,k}, \tag{15}$$

where V_i^k is given by (2) and (3). From (4), (6) and (14), we obtain

$$X = C - 2\frac{h}{S}D, \quad N = \frac{2}{S}D.$$

Taking the derivative of these expressions and using (15), it follows from a straightforward computation that $N_{,i} = \sum_{j=1}^{2} W_{ij}X_{,j}$, where

$$W = 2V(SI - 2hV)^{-1}$$

and V is defined by (2) and (3). From matrix properties it follows that W is given by (7).

□

We observe that, when the matrix V is diagonal, the parametrization X in the theorem above, is a Ribaucour transformation of Y with respect to $\frac{Y_{,i}}{|Y_{,i}|}$, (see [CT]).

2. BGW-surfaces in $\mathbb{H}^3$.

In this section, we consider $\mathbb{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$ as the model of the *upper half-space*, with the metric $ds^2 = \frac{1}{x_3^2}(dx_1^2 + dx_2^2 + dx_3^2)$ and *ideal boundary* $\mathbb{C}_\infty = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\} \cup \{\infty\}$. Now, let M be a connected orientable Riemann surface, $X : M \to \mathbb{H}^3$ an immersion with Gauss map η and *hyperbolic*

Gauss map G. That is, for any $p \in M$, the oriented geodesic emanating from $X(p)$ meets the ideal boundary $\mathbb{C}_\infty$ at $G(X(p))$. The hyperbolic Gauss map defines a sphere congruence, for which $X(M)$ and $G(X(M))$ are envelopes and the radius of each sphere define the *radius function* h. Then $X : M \to \mathbb{H}^3$ is called a *generalized Weingarten surface of Bryant type* (in short, BGW-surface) if the mean curvature H, the Gauss curvature K_I and radius function h satisfy

$$2ach^{\frac{2(c-1)}{c}}(H-1) + (a+b-ach^{\frac{2(c-1)}{c}})K_I = 0, \tag{16}$$

for some $a, b, c \in \mathbb{R}$, $a+b \neq 0$ and $c \neq 0$.

Lemma 2.1: *Let* M *be a connected orientable Riemann surface. Then* $X : M \to \mathbb{H}^3$ *is an immersion if, and only if, there exist functions* $g : M \to \mathbb{C}_\infty$ *holomorphic and* $h : M \to \mathbb{R}^+_\infty$, *such that* $X(M)$ *is locally parametrized by*

$$X(z) = (g, 0) - \frac{2h}{S}\left(\frac{g'}{|g'|^2}(h_{,1} + ih_{,2}), -1\right), \tag{17}$$

where $z = u_1 + iu_2 \in \mathbb{C}$, $\mathbb{R}^+_\infty = \mathbb{R}^+ \cup \{+\infty\}$, $g'(z) \neq 0$ *everywhere,*

$$S = \frac{(h_{,1})^2 + (h_{,2})^2}{|g'|^2} + 1. \tag{18}$$

Moreover, the normal Gauss map is given by

$$\eta(z) = \frac{2h}{S}\left[(0,0,1) + \frac{2}{S}\left(\frac{g'}{|g'|^2}(h_{,1} + ih_{,2}), -1\right)\right],$$

the Weingarten matrix is

$$\widetilde{W} = \frac{2}{P}[2hV + (S - 2h\, tr(V))I] - I, \tag{19}$$

where

$$P = S^2 - 2hS\, tr(V) + 4h^2\, det(V) \neq 0, \tag{20}$$

$$V_{ij} = \frac{1}{|g'|^2}\left(h_{,ij} - \sum_{\ell=1}^{2} \Gamma^\ell_{ij} h_{,\ell}\right), \quad 1 \leq i, j \leq 2, \tag{21}$$

with

$$\Gamma^i_{ii} = \frac{|g'|_{,i}}{|g'|}, \qquad \Gamma^i_{ij} = \frac{|g'|_{,j}}{|g'|} = -\Gamma^j_{ii}, \quad 1 \leq i \neq j \leq 2. \tag{22}$$

The mean curvature H and Gauss curvature K_I are given by

$$H = \frac{2}{P}\left[\frac{h \triangle h}{|g'|^2} - S\right] + 1, \quad K_I = \frac{4}{P}\left[\frac{h \triangle h}{|g'|^2} - S + 1\right]. \tag{23}$$

Proof. $X : M \to \mathbb{H}^3$ is an immersion if, and only if, $X(M)$ is the envelope of a sphere congruence, whose other envelope is contained in $\mathbb{C}_\infty$. By Theorem 1.1, there exist functions $g : M \to \mathbb{C}_\infty$ holomorphic and $h : M \to \mathbb{R}^+_\infty$, such that $Y(z) = (g(z), 0)$ and $L_{ii} = |g'|^2$. Let $X = (X_1, X_2, X_3)$ and $N = (N_1, N_2, N_3)$ be defined by (4)-(6), then the Gauss map is given by $\eta = X_3 N$. Using the connection induced by the metric of $\mathbb{H}^3$, it follows from a straightforward computation that the Weingarten matrix of X in $\mathbb{H}^3$ is

$$\widetilde{W} = X_3 W - N_3 I,$$

where W is given by (7). Using (2)-(8), we obtain (17)-(22). Moreover, since $H = -\frac{1}{2}\mathrm{tr}(\widetilde{W})$ and $K_I = \det(\widetilde{W}) - 1$, the mean curvature and Gauss curvature are given by (23).

□

Theorem 2.2: *Let M be a connected orientable Riemann surface. Then $X : M \to \mathbb{H}^3$ is a flat surface if, and only if, there exist holomorphic functions $f, g : M \to \mathbb{C}_\infty$, such that $X(M)$ is locally parametrized by*

$$X(z) = (g, 0) - \frac{2h}{T}(hg'\overline{f'}, -|g'|^2),$$

where $g'(z) \neq 0$ everywhere,

$$h = e^{\mathrm{Re}(f)},$$

$\mathrm{Re}(f)$ *is the real part of f,*

$$T = h^2|f'|^2 + |g'|^2,$$

$$-T^2 + 2T|g'|^2 + 4h^2|g'|^4 \det(V) \neq 0,$$

and the matrix V *is given by* (21) *and* (22).

Proof. From Lemma 2.1 taking $h = e^{\phi}$ and using (18) and (23) we have that X is a flat surface if, and only if, $\triangle\phi = 0$. Therefore there exists a holomorphic function f such that $\phi = \mathrm{Re}(f)$, $h = e^{\mathrm{Re}(f)}$ and

$$h_{,1} + ih_{,2} = h\overline{f'}.$$

Then, from (17), (18) and (20) the result follows.

□

The method developed above is based on considering a flat surface in $\mathbb{H}^3$, as the envelope of a sphere congruence whose other envelope is contained in $\mathbb{C}_\infty$, with the radius function determined by a holomorphic function. Another method, presented in [GMM2], gives a conformal representation of flat surfaces using the complex structure induced by its second fundamental form. In [Ro] an alternative derivation is given for a local Weierstrass representation of flat surfaces, which uses as a starting point an old result due to Luigi Bianchi. Roughly speaking, that result says that the correspondence between the two hyperbolic Gauss maps of a surface $M \subset \mathbb{H}^3$ is weakly conformal if, and only if, M is flat.

A direct application of Theorem 2.2 yields some interesting examples:

Example 2.3: Let $\lambda \neq 0$ be a real constant, $M = \mathbb{C} - \{(0,0)\}$, $f(z) = ln(\lambda^2 z^2)$ and $g(z) = z$. Then the maps $X_\lambda : \mathbb{C} - \{(0,0)\} \to \mathbb{H}^3$ given by

$$X_\lambda(z) = (z,0) - \frac{2\lambda^2|z|^2}{4\lambda^4|z|^2 + 1}(2\lambda^2 z, -1)$$

define a one-parameter family of complete flat surfaces of revolution.

Example 2.4: Let $\lambda \neq 0$ be a real constant, $M = \mathbb{C}$, $f(z) = \lambda z$ and $g(z) = z$. Then the maps $X_\lambda : \mathbb{C} \to \mathbb{H}^3$ given by

$$X_\lambda(z) = (z,0) - \frac{2e^{\lambda u_1}}{\lambda^2 e^{2\lambda u_1} + 1}(e^{\lambda u_1}\lambda, 0, -1)$$

define a one-parameter family of complete flat ruled surfaces.

A useful characterization of BGW-surfaces can be obtained through their relation to solutions of Liouville's equation. In order to do that, we will need the following result whose proof and further details can be found in [Li].

Theorem 2.5: *A global solution of the Liouville equation*

$$\triangle\nu = Ae^{-2\nu}$$

is expressed by

$$e^{\nu(u_1,u_2)} = \frac{1+A|f(z)|^2}{2|f'(z)|}, \quad z = u_1 + iu_2 \in \mathbb{C} \tag{24}$$

where f is a global meromorphic function, such that $f'(z) \neq 0$ everywhere and it admits only simple poles. Moreover, f and its transformations $\frac{d_1 f - \overline{d_2}}{d_2 f - \overline{d_1}}$, $|d_1|^2 + |d_2|^2 > 0$, give all the possibilities for (24) to hold.

We observe that, for each solution ν of the Liouville equation, the metric $ds^2 = e^{-2\nu}((du_1)^2 + (du_2)^2)$, being conformal to the Euclidean metric, have constant Gauss curvature A.

Theorem 2.6: *Let M be a connected orientable Riemann surface. Then $X : M \to \mathbb{H}^3$ is a BGW-surface if, and only if, there exist holomorphic functions $f, g : M \to \mathbb{C}_\infty$, such that $X(M)$ is locally parametrized by*

$$X(z) = (g,0) - \frac{h}{S|f'|^3|g'|^3}(ch^{\frac{(c-1)}{c}} f'(g')^2\Upsilon, -2|f'|^3|g'|^3), \tag{25}$$

where $|f'|^2 + |g'|^2 \neq 0$, $(1 + A|f|^2)^c > 0$ everywhere, with $A = a/(a + b)$,

$$h = \left[|g'|\frac{(1+A|f|^2)}{2|f'|}\right]^c, \tag{26}$$

$$S = \frac{c^2 h^{\frac{2(c-1)}{c}}}{4|f'|^4|g'|^2}|\Upsilon|^2 + 1, \tag{27}$$

$$\Upsilon = [1 + A|f|^2](\overline{f'g''} - \overline{g'f''}) + 2Af(\overline{f'})^2\overline{g'}, \tag{28}$$

and

$$-S^2 + 2S(1 - Ach^{\frac{2(c-1)}{c}}) + 4h^2 \det(V) \neq 0,$$

with the matrix V given by (21) *and* (22).

Proof. From Lemma 2.1 taking $h = e^{c\phi}$, $c \neq 0$ a real constant and using (18), (23) and (16), we have that X is a BGW-surface if, and only if, $\triangle\phi = A|g'|^2 e^{-2\phi}$ and $A = a/(a+b)$. Since $\triangle \ln(|g'|) = 0$, we have $\phi = \ln(|g'|) + \nu$, where $\triangle\nu = Ae^{-2\nu}$. From Theorem 2.5, it follows that the solutions to this equation are given by (24). Therefore h is given by (26) and

$$h_{,1} + ih_{,2} = \frac{c}{2|f'|^3|g'|}\left[\frac{|g'|(1+A|f|^2)}{2|f'|}\right]^{(c-1)} f'g'\Upsilon,$$

where Υ is given by (28). Using (18) we obtain that S is given as in (27) and from (17) the expression for X follows.

□

The method detailed above consists of looking at a BGW-surface in $\mathbb{H}^3$, as the envelope of a sphere congruence whose envelope is contained in $\mathbb{C}_\infty$, with the radius function determined by a solution of Liouville equation. When $c = 1$, the surfaces obtained are BLW-surfaces and in order to appreciate how this method differs from others in the literature it will be useful to look at some particular cases. For instance [GMM2] gives a complex representation of a BLW-surface M, induced by a Riemannian metric $\sigma = aI + bII$ where I and II are the first and second fundamental forms of M. This complex structure is such that its hyperbolic Gauss map is a conformal map. In particular, when $b = 0$ we get Bryant surfaces and [Br], [GG], [LR], [Sm], [ST2] and [UY1] introduce a complex structure such that its hyperbolic Gauss map is a conformal map. hand, when $a = 0$ we obtain flat surfaces and the method here becomes analogous, to the one in Theorem 2.2 and both are different from the one used in [GMM2] and [Ro].

Remark 2.7: For later convenience, from Theorem 2.6, using (26) and (27), when $c \geq 1$ the parametrization (25), can be written as

$$\begin{aligned} X(z) = \;& (g - \tfrac{2c}{T_1}|g'|^{2(c-1)}[1 + A|f|^2]^{(2c-1)} f'(g')^2 \Upsilon, \\ & \tfrac{2^{(c+1)}}{T_1}|f'g'|^{(c+2)}[1 + A|f|^2]^c), \end{aligned} \tag{29}$$

with Υ given by (28) and

$$T_1 = c^2(|g'|\,[1 + A|f|^2])^{2(c-1)}|\Upsilon|^2 + 2^{2c}|f'|^{2(c+1)}|g'|^2. \tag{30}$$

On the other hand, when $c < 1$, (25) becomes

$$\begin{aligned} X(z) = \;& (g - \tfrac{c2^{(3-2c)}}{T_2}|f'|^{2(1-c)}[1 + A|f|^2] f'(g')^2 \Upsilon, \\ & \tfrac{2^{(3-c)}}{T_2}|f'g'|^{(4-c)}(|g'|\,[1 + A|f|^2])^{(2-c)}), \end{aligned} \tag{31}$$

with

$$T_2 = c^2(2|f'|)^{2(1-c)}|\Upsilon|^2 + 4|f'|^4|g'|^{2(2-c)}[1 + A|f|^2]^{2(1-c)}. \tag{32}$$

In order to study the asymptotic behavior of a BGW-surface at $\mathbb{C}_\infty$, we define the asymptotic boundary of a set $\Sigma \in \mathbb{H}^3$ as

$$\partial_\infty(\Sigma) = \mathrm{cl}(\Sigma) \cap \mathbb{C}_\infty$$

where $\mathrm{cl}(\Sigma)$ is the closure of Σ in $\{x_3 \geq 0\} \cup \{\infty\}$.

When $A > 0$ and $c \geq 1$, the geometry of the surface is similar to that of a surface with constant mean curvature $H = 1$. On the other hand, when $A < 0$ the following result holds:

Proposition 2.8: *Let $\widetilde{M}$ be a connected Riemann surface, $f, g : \widetilde{M} \to \mathbb{C}_\infty$ holomorphic functions and $A < 0$. Suppose*

$$M_A = \{p \in \widetilde{M}/1 + A|f(p)|^2 > 0\}, \quad \Gamma_A = \{p \in \widetilde{M}/1 + A|f(p)|^2 = 0\},$$

are non empty subsets of $\widetilde{M}$, let M be a non empty connected subset of M_A and $\Gamma = \partial M$ a non empty subset of Γ_A. Then the maps $X : M \to \mathbb{H}^3$ given by Theorem 2.6 are BGW-surfaces and

$$g(\Gamma) \subset \partial_\infty(M).$$

Proof. Using Theorem 2.6, (29) and (30) we have, for each $w \in \Gamma$,

$$\lim_{z \to w} T_1(z) = 2^{2c}|f'(w)|^{2(c+2)}|g'(w)|^2 \neq 0, \quad c \geq 1$$

and $\lim_{z \to w} X(z) = g(w)$. Analogously, using (31) and (32) we have, for each $w \in \Gamma$,

$$\lim_{z \to w} T_2(z) = -2^{(3-2c)}Ac^2|f'(w)|^{2(2-c)}|f(w)||g'(w)| \neq 0, \quad c < 1, c \neq 0$$

and $\lim_{z \to w} X(z) = g(w)$.

□

3. Applications.

As a result of the method developed in the previus section, we obtain four one-parameter families of complete BGW-surfaces in $\mathbb{H}^3$. We start by replacing $A = a/(a+b)$ in (16) to obtain

$$2Ach^{\frac{2(c-1)}{c}}(H-1) + (1 - Ach^{\frac{2(c-1)}{c}})K_I = 0. \tag{33}$$

Proposition 3.1: *Let $A \neq 0$, $c \geq 1$ and $\lambda > 0$ be real constants such that, if $A > 0$, then $c \in \mathbb{N}$ and $c > 1$. Consider the set $M_{\lambda A} = \{z \in \mathbb{C}/1+A\lambda^2|z|^2 > 0\}$. Then the maps $X_\lambda : M_{\lambda A} \to \mathbb{H}^3$ given by*

$$X_\lambda(z) = (z - \frac{cA}{T(z)}(1 + A\lambda^2|z|^2)^{(2c-1)}z, \frac{2^{(c-1)}}{T(z)}\lambda^{(c-2)}(1 + A\lambda^2|z|^2)^c)$$

with

$$T(z) = c^2(1 + A\lambda^2|z|^2)^{2(c-1)}\lambda^2A^2|z|^2 + 2^{2(c-1)}\lambda^{2(c-1)},$$

define a one-parameter family of complete BGW-surfaces of revolution, whose mean curvature H and Gaussian curvature K_I satisfy (33).

Proof. From Theorem 2.6 and Remark 2.7, taking $g(z) = z$, $f(z) = \lambda z$ and using (29), (30) and (28), we obtain the expression for X_λ given above. Taking

$u_1 = r\cos\theta$ and $u_2 = r\sin\theta$, we have that X_λ is a surface of revolution, with the x_3 axis as the rotation axis and the generating curve, in the x_2x_3 plane, given by

$$\begin{aligned}\alpha(r) = \ & \tfrac{1}{T(r)}(r[cA(1+A\lambda^2r^2)^{(2c-1)}(-1+(c-1)A\lambda^2r^2) \\ & +(2\lambda)^{2(c-1)}], 2(2\lambda)^{(c-2)}(1+A\lambda^2r^2)^c)\end{aligned}$$

where

$$T(r) = c^2(1+A\lambda^2r^2)^{2(c-1)}\lambda^2A^2r^2 + 2^{2(c-1)}\lambda^{2(c-1)}.$$

Since $x_2(r)$ is an odd function and $x_3(r)$ is even, the curve is symmetrical with respect to the x_3 axis. Since $x_3'(0) = 0$ the curve is regular at $\alpha(0)$. When $A > 0$ we have $M_{\lambda A} = \mathbb{C}$, in which case $r \in \mathbb{R}$ and $\lim\limits_{r\to\infty} x_2(r) = \infty$. Therefore $X_\lambda(\mathbb{C})$ is a closed subset in $\mathbb{H}^3$ and X_λ is complete. When $A < 0$ we have $M_{\lambda A} = \{z \in \mathbb{C}/|z| < \frac{1}{\lambda\sqrt{-A}}\}$ and, from Proposition 2.8, $X_\lambda(M_{\lambda A})$ is a closed subset of $\mathbb{H}^3$, so X_λ is complete.

□

It follows from the above Proposition, that when $c = 1$, the surfaces are BLW-surfaces.

Proposition 3.2: *Let $A \neq 0$, $c < 1$ and $\lambda > 0$ be real constants such that if $A > 0$, then $c \in \mathbb{N}$. Consider the set $M_{\lambda A} = \{z \in \mathbb{C}/1 + A\lambda^2|z|^2 > 0\}$. Then the maps $X_\lambda : M_{\lambda A} \to \mathbb{H}^3$ given by*

$$X_\lambda(z) = (z - \frac{cA}{T(z)}2^{2(1-c)}\lambda^{2(1-c)}(1+A\lambda^2|z|^2)z, \frac{2^{(1-c)}}{T(z)}\lambda^{(-c)}(1+A\lambda^2|z|^2)^{(2-c)})$$

with

$$T(z) = c^2(2\lambda)^{2(1-c)}A^2\lambda^2|z|^2 + (1+A\lambda^2|z|^2)^{2(1-c)},$$

defines a one-parameter family of complete BGW-surfaces of revolution, whose mean curvature H and Gaussian curvature K_I satisfy the relation (33).

Proof. From Theorem 2.6 and Remark 2.7, taking $g(z) = z$, $f(z) = \lambda z$ and using (31), (32) and (28), we obtain the expression for X_λ given above. Taking

$u_1 = r\cos\theta$ and $u_2 = r\sin\theta$, we have that X_λ is a surface of revolution, with the x_3 axis as the rotation axis and the generating curve, in the x_2x_3 plane, given by

$$\alpha(r) = (r - \frac{cA}{T(r)} 2^{2(1-c)}\lambda^{2(1-c)}(1 + A\lambda^2 r^2)r, \frac{2^{(1-c)}}{T(r)}\lambda^{(-c)}(1 + A\lambda^2 r^2)^{(2-c)})$$

where

$$T(r) = c^2(2\lambda)^{2(1-c)}A^2\lambda^2 r^2 + (1 + A\lambda^2 r^2)^{2(1-c)}.$$

Since $x_2(r)$ is an odd function and $x_3(r)$ is even the curve is symmetrical with respect to the x_3 axis. Since $x_3'(0) = 0$ the curve is regular at $\alpha(0)$. When $A > 0$ we have $M_{\lambda A} = \mathbb{C}$, in which case $r \in \mathbb{R}$ and $\lim\limits_{r\to\infty} x_2(r) = \infty$. Therefore $X_\lambda(\mathbb{C})$ is a closed subset in $\mathbb{H}^3$ and X_λ is complete. When $A < 0$ we have $M_{\lambda A} = \{z \in \mathbb{C}/|z| < \frac{1}{\lambda\sqrt{-A}}\}$ and from Proposition 2.8, $X_\lambda(M_{\lambda A})$ is a closed subset of $\mathbb{H}^3$. Therefore X_λ is complete.

$\square$

Proposition 3.3: *Let $A \neq 0$, $c \geq 1$ and $\lambda > 0$ be real constants such that $A\lambda^2 > -1$ when $A < 0$. Consider the set $M_{\lambda A} = \{z \in \mathbb{C}/1 + A\lambda^2 e^{2u_1} > 0\}$. Then the maps $X_\lambda : M_{\lambda A} \to \mathbb{H}^3$ given by*

$$X_\lambda(z) = (z + \frac{2c}{T(z)}(1 + A\lambda^2 e^{2u_1})^{(2c-1)}(1 - A\lambda^2 e^{2u_1}), \frac{2^{(c+1)}}{T(z)}(\lambda e^{u_1})^c(1 + A\lambda^2 e^{2u_1})^c)$$

with

$$T(z) = c^2(1 + A\lambda^2 e^{2u_1})^{2(c-1)}(1 - A\lambda^2 e^{2u_1})^2 + (2\lambda e^{u_1})^{2c},$$

define a one-parameter family of complete ruled BGW-surfaces, whose mean curvature H and Gaussian curvature K_I satisfy (33).

Proof. From Theorem 2.6 and Remark 2.7, taking $g(z) = z$, $f(z) = \lambda e^z$ and using (29), (30) and (28), we obtain the expression for X_λ given above. Then X_λ is a ruled surface, whose rulings are parallel to the x_2 axis and the curve directrix, in the x_1x_3 plane, is given by

$$\alpha(u_1) = (u_1 + \frac{2cA}{T(z)}(1 + A\lambda^2 e^{2u_1})^{(2c-1)}(1 - A\lambda^2 e^{2u_1}), \frac{2^{(c+1)}}{T(z)}(\lambda e^{u_1})^c(1 + A\lambda^2 e^{2u_1})^c)$$

with

$$T(u_1) = c^2(1 + A\lambda^2 e^{2u_1})^{2(c-1)}(1 - A\lambda^2 e^{2u_1})^2 + (2\lambda e^{u_1})^{2c}.$$

When $A > 0$ we have $M_{\lambda A} = \mathbb{C}$, in which case $u_1 \in \mathbb{R}$ and $\lim_{r\to\pm\infty} x_1(r) = \pm\infty$. Therefore $X_\lambda(\mathbb{C})$ is a closed subset in $\mathbb{H}^3$ and X_λ is complete. When $A < 0$ we have $M_{\lambda A} = \{z \in \mathbb{C}/|u_1| < \ln(\frac{1}{\lambda\sqrt{-A}})\}$ and from Proposition 2.8, $X_\lambda(M_{\lambda A})$ is a closed subset of $\mathbb{H}^3$ so X_λ is complete.

□

It follows from the above Proposition, that, when $c = 1$, the surfaces are BLW-surfaces.

Proposition 3.4: *Let $A \neq 0$, $c < 1$ and $\lambda > 0$ be real constants, such that $A\lambda^2 > -1$, when $A < 0$,. Consider the set $M_{\lambda A} = \{z \in \mathbb{C}/1 + A\lambda^2 e^{2u_1} > 0\}$. Then the maps $X_\lambda : M_{\lambda A} \to \mathbb{H}^3$ given by*

$$\begin{aligned} X_\lambda(z) = \ & (z + \tfrac{2c}{T(z)}(2\lambda e^{u_1})^{2(1-c)}(1 + A\lambda^2 e^{2u_1})(1 - A\lambda^2 e^{2u_1}), \\ & \tfrac{2^{(3-c)}}{T(z)}(\lambda e^{u_1}(1 + A\lambda^2 e^{2u_1}))^{(2-c)}) \end{aligned}$$

with

$$T(z) = c^2(2\lambda e^{u_1})^{2(1-c)}(1 - A\lambda^2 e^{2u_1})^2 + 4(\lambda e^{u_1})^2(1 + A\lambda^2 e^{2u_1})^{2(1-c)},$$

define a one-parameter family of complete ruled BGW-surfaces, whose mean curvature H and Gaussian curvature K_I satisfy (33).

Proof. From Theorem 2.6 and Remark 2.7, taking $g(z) = z$, $f(z) = \lambda e^z$ and using (31), (32) and (28), we obtain the expression for X_λ given above. Besides X_λ is a ruled surface, whose rulings are parallel to the x_2 axis and with the curve directrix, in the x_1x_3 plane, given by

$$\begin{aligned} \alpha(u_1) = \ & (u_1 + \tfrac{2c}{T(z)}(2\lambda e^{u_1})^{2(1-c)}(1 + A\lambda^2 e^{2u_1})(1 - A\lambda^2 e^{2u_1}), \\ & \tfrac{2^{(3-c)}}{T(z)}(\lambda e^{u_1}(1 + A\lambda^2 e^{2u_1}))^{(2-c)}) \end{aligned}$$

with

$$T(u_1) = c^2(2\lambda e^{u_1})^{2(1-c)}(1 - A\lambda^2 e^{2u_1})^2 + 4(\lambda e^{u_1})^2(1 + A\lambda^2 e^{2u_1})^{2(1-c)}.$$

When $A > 0$ we have $M_{\lambda A} = \mathbb{C}$. In this case $u_1 \in \mathbb{R}$ and $\lim\limits_{r\to\pm\infty} x_1(r) = \pm\infty$. Therefore $X_\lambda(\mathbb{C})$ is a closed subset of $\mathbb{H}^3$ and X_λ is complete. When $A < 0$ we have $M_{\lambda A} = \{z \in \mathbb{C}/|u_1| < \ln(\frac{1}{\lambda\sqrt{-A}})\}$ and from Proposition 2.8 we have that $X_\lambda(M_{\lambda A})$ is a closed subset of $\mathbb{H}^3$. Therefore X_λ is complete.

□

This method would also be suitable for other constructions, in more particular cases, e. g., $c = 1$, but there is already a vast literature dealing with such examples, as well as their geometric properties, a few of them already referenced to throughout this paper.

On the other hand, the more general cases ($c \neq 1$), the study of basic problems such as existence, symmetry, behaviour at infinity, global structure and classification, present a promising open field for future research.

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