Preface

It is heartwarming to see the enthusiasm of so many mathematicians who enthusiastically contributed their research to these volumes in honor of Manfredo Do Carmo's 80'th birthday.

Manfredo created fields of investigation by brilliant innovative ideas. He (and others) then developed these ideas to make profund contributions to geometry. From time to time, Manfredo would also discover isolated beautiful theorems.

I will try to give a panorama of his work. Since this is extensive and began almost 50 years ago, this will not be complete.

Manfredo's thesis of 1963 proved a pinching theorem for Kähler Manifolds: if the Kählerian curvature is between 0.8 and 1, then the manifold has the same cohomology as $\mathbb{CP}^n$. This paper was published in the Annals of Mathematics and Manfredo did not pursue this subject further [10]. Shortly after Manfredo's work, Klingenberg generalized this theorem.

A central theme in many of his papers is to understand the global geometry of submanifolds of a Riemannian manifold in terms of its second fundamental form.

In collaboration with Elon Lima [14] and independently with Warner [16], Manfredo generalized Hadamard's theorem on locally convex hypersurfaces of $\mathbb{R}^{n+1}$.

Manfredo and Warner proved that a compact surface in $\mathbb{S}^3$, whose extrinsic curvature is non-negative is an embedded sphere.

With Lima [14], Manfredo proved that an immersion of a hypersurface in $\mathbb{R}^{n+1}, n \geq 2$, whose second fundamental form is semi-definite, and definite at least one point, is embedded. This was also proved and generalized by Sacksteder.

In 1968, Manfredo began studying minimal submanifolds, and he has continued to do so (very fortunately for the subject) throughout his career.

Together with Nolan Wallach, he wrote 3 papers on minimal submanifolds of spheres [17,18,19]. They classified those that are isometric immersions of spheres of curvature $c > 0$. In particular, they gave many more examples than those conjectured by Chern. They described the moduli space as a convex set with stratified boundary. The structure of this boundary was studied by many mathematicians in future work, in particular, by Wolfgang Ziller.

In 1968, together with Chern and Kobayashi [9], Manfredo wrote an important paper on minimal submanifolds M^n of $\mathbb{S}^{n+p}$, whose second fundamental form has constant norm C. Simons gap theorem says there are no such examples for $0 < C < n$. Chern, Kobayashi and Manfredo narrowed the gap: there are no examples for $0 < C < \dfrac{n}{2-1/p}$. Moreover, if $C = \dfrac{n}{2-1/p}$, then M is a Clifford torus $\mathbb{S}^k \times \mathbb{S}^{n-k}$, or M is a Veronese surface in $\mathbb{S}^4$. This is an often quoted theorem.

In 1974, Manfredo and Lucas Barbosa began to study stable minimal surfaces. Their theorems of this period have been of major importance in the subject until today. Few theorems in mathematics are cited more than ten years after their publication; this work of Manfredo and Lucas is an exception.

They proved that an orientable minimal surface in $\mathbb{R}^3$ whose Gaussian image has area less than 2π is stable [7]. This is a fundamental tool to study minimal surfaces globally. For example, stability implies curvature estimates (at a fixed distance from the boundary) and bounded curvature enables one to understand limits of such surfaces.

In 1979, Manfredo and Peng [20] proved that a complete, orientable and stable mininal surface in $\mathbb{R}^3$ is a plane (it was proved independently by D. Fischer-Colbrie and R. Schoen).

I suspect I used this theorem in most of the papers I wrote these past 20 years. Here is one reason this theorem is so important. When studying the global geometry of minimal and constant mean curvature surfaces in 3-manifolds, one begins by controlling the second fundamental form (the curvature). When a sequence of points has curvature diverging, one dilates the ambient metric in fixed neighborhoods of the points and puts the resulting balls in $\mathbb{R}^3$ (using appropriate coordinates) to obtain a sequence of compact surfaces in $\mathbb{R}^3$, with bounded (extrinsic) curvature, curvature one at the origin, and mean curvature converging to zero. A subsequence of these surfaces converges to a complete minimal surface (with curvature one at the origin). If the original surface we were studying were stable, this limit minimal surface in $\mathbb{R}^3$ would be stable. Hence it would be a plane by the Do Carmo-Peng, Schoen theorem. This contradicts curvature one at the origin. Therefore the original surface has bounded curvature (when it is stable).

Lucas and Manfredo [6] continued their study of stability: they proved that a minimal disk D in $\mathbb{R}^n$ whose total curvature is less than $\frac{4}{3}\pi$ is stable.

They then began the study of stable constant mean curvature surfaces [5]. First, they proved that geodesic spheres are the unique stable hypersuperfaces in $\mathbb{R}^n$. Then, together with Eschenburg [8], they extended this result to space forms. They began the study of stability with respect to other curvature functions. Manfredo, Alencar and Colares [3] considered scalar curvature: they proved that a constant scalar curvature compact hypersurface of a space form, is a geodesic sphere. In $\mathbb{S}^n$ one need suppose the hypersurface is contained in a hemisphere.

In 1983, Manfredo and Blaine Lawson [13] wrote a seminal paper on properly embedded H-surfaces Σ in $\mathbb{H}^3$ (I was particularly inspired by this paper). They studied the surface Σ in terms of its asymptotic boundary at infinity $\partial_\infty(\mathbb{H}^3)$. They proved that Σ is a horosphere when $\partial_\infty(\Sigma)$ is one point and Σ is an equidistant totally umbilic surface when $\partial_\infty(\Sigma)$ is a circle. Subsequently, much research has been done on this subject.

In 1985, Manfredo began working on another subject. Together with Marcos Dajczer and Francisco Mercuri [12], they classified conformally flat hypersurfaces in space forms of dimension greater than or equal to five.

They describe them as built from submanifolds foliated by $n-1$ spheres joined together along their boundaries by three types of umbilic submanifols which they describe. Their technique is called conformal surgery.

In 1987, Manfredo and Dajczer [11] generalized rigidity results on hypersurfaces of $\mathbb{R}^{n+1}$ to higher codimension.

They gave a condition (inequality) on the conformal nullity of a conformal immersion $M^n \to \mathbb{R}^{n+k}, k \leq 4$, that implies conformal rigidity. They also gave a condition on the nullity to guaranty rigidity of isometric immersions $M^n \to \mathbb{R}^{n+k}$ when $n > 2k, k \leq 5$.

Now returning to minimal and constant mean curvature submanifolds, Manfredo realized the importance of those that are not stable but admit a finite dimensional space of deformations that reduce the volume; this means finite index.

In 1993, Manfredo and Hilário Alencar [2] studied complete, non-compact, hypersurfaces Σ^n of N^{n+1} of constant mean curvature H. They found an estimate for H assuming Σ has polynomial volume growth and finite index.

As a consequence of this, they proved that $H = 0$ when $\mathrm{Ric}(N) \geq 0$, and if $\mathrm{Ric}(N) \geq -\delta$, then $H^2 \leq \delta$. Manfredo, Hilário and Walcy Santos [4] proved a gap theorem for compact hypersurfaces of scalar curvature one in spheres and they characterized the extremal case. Then, they proved if $\Sigma \subset \mathbb{RP}^n$ is a closed, orientable hypersurface of scalar curvature one, then the index is greater than one and equal to one if and only if it is the projection of a Clifford Torus in $\mathbb{S}^n$.

In 1994, Manfredo and Hilário [1] extended Simons gap theorem (for minimal hypersurfaces) to compact constant mean curvature H hypersurfaces M of the sphere $\mathbb{S}^{n+1}$. Let ϕ be the trace free part of the shape operator of M (so ϕ is the shape operator when $H = 0$). There is a positive number $B(H, n) = B$ such that when $|\phi|^2 \leq B$, one can classify M; B is the positive real root of a certain quadratic polynomial and $B = n^2$ when $H = 0$. Their theorem states that if $|\phi|^2 \leq B$, then $|\phi| = 0$ or $|\phi|^2 = B$. When $|\phi| = 0$ then M is totally umbilical. When $|\phi|^2 = B$ then (a) $H = 0$ and M is a Clifford torus; (b) $H \neq 0$, $n \geq 3$ and M is an $H(r)$-torus with $r^2 < \dfrac{n-1}{n}$; (c) $H \neq 0$, $n = 2$ and M is an $H(r)$-torus with $r^2 \neq \dfrac{n-1}{n}$.

In the same spirit, Manfredo and Zhou Detang [21] obtained sharp estimates on the first eigenvalue of complete, non-compact, Riemannian manifolds under certain volume growth assumptions. They apply this to obtain estimates of the mean curvature of hypersurfaces.

In 2000, Manfredo, Antonio Ros and Manoel Ritoré [15] proved a beautiful theorem: the orientable, compact, minimal hypersurfaces of $\mathbb{RP}^n$ of index one are immersed totally geodesic spheres or a quotient of a Clifford Torus.

Manfredo has done a great deal of important work on which I will not comment. I hope the reader - and specially, Manfredo - will pardon me.

I will finish with a paper I like very much; It is the last paper Manfredo has written and it will be published in the Journal of Differential Geometry.

Manfredo, Alencar and Tribuzy study surfaces M of parallel mean curvature in $\mathcal{Q} \times \mathbb{R}$, where $\mathcal{Q}$ is an n-dimensional space form. They describe very well the geometry of these surfaces.

When M is topologically a sphere, they completely classify the geometry. This is a generalized Hopf theorem. When $\mathcal{Q}$ has dimension 4, this classification gives previously unknown rotational examples. They define a "Hopf-type" holomorphic quadratic differential on such a surface M and they do very ingenious arguments to prove their theorems. Also ,when the mean curvature of M is parallel and M is complete with non-negative Gaussian curvature, they classify the geometry.

Manfredo, I am very privileged to have known you all these (almost 50) years, and I wish you a very happy 80'th birthday.

Harold Rosenberg

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