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NEW ESTIMATES FOR THE SCALAR CURVATURE OF COMPLETE MINIMAL HYPERSURFACES IN $\mathbb{S}^4$

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Dedicated to Professor Manfredo do Carmo on the occasion of his 80th birthday

Abstract

Let M^3 be a complete minimal hypersurface immersed in the unit sphere $\mathbb{S}^4$. In this paper, starting from hypotheses on the Gauss-Kronecker curvature we obtain estimates for the scalar curvature of M^3.

1 Introduction

Denote by $\mathbb{S}^N$ the N-dimensional unit sphere in $\mathbb{R}^{N+1}$. Let M^n be an n-dimensional submanifold minimally immersed in $\mathbb{S}^{n+p}$. Denote by R the scalar curvature of M^n and by S the square of the length of the second fundamental form of M^n. In his celebrated paper, J. Simons [6] obtained the following inequality for the Laplacian of S

$$\frac{1}{2}\Delta S \geq S\left(n-\left(2-\frac{1}{p}\right)S\right). \tag{1.1}$$

As an application of (1.1), Simons proved that if M^n is closed then either M^n is totally geodesic, or $S=\dfrac{n}{2-1/p}$, or $\sup S > \dfrac{n}{2-1/p}$. In this paper we prove an inequality similar to that of Simons given above for complete minimal hypersurfaces in $\mathbb{S}^4$.

Theorem 1.1. *Let M^3 be a complete minimal hypersurface in $\mathbb{S}^4$. Let K be the Gauss-Kronecker curvature of M^3. If S is bounded and $|K|$ is bounded away from zero, then* $\inf S \leq 3 \leq \sup S$.

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The inequality $\sup S \geq 3$ is a particular case of one stablished by Cheng in [1] that extended Simons' result, for complete submanifolds. We point out that, although for $p = 1$ the sharp estimate $S \geq n$ was due to Simons, the characterization of the hypersurfaces satisfying $S = n$ was obtained independently by Chern, Do Carmo and Kobayashi [2] and Lawson [3]. Up to now, it is not known if there exist complete minimal hypersurfaces satisfying $\sup S = n$ and that are not congruent to the Clifford tori $\mathbb{S}^k\left(\sqrt{\frac{k}{n}}\right) \times \mathbb{S}^{n-k}\left(\sqrt{\frac{n-k}{n}}\right)$.

By the fact that $R = 6 - S$, in case $n = 3$, see (2.4), we immediately obtain the following consequence of Theorem 1.1.

Corollary 1.1. *Let M^3 be a complete minimal hypersurface in $\mathbb{S}^4$. If R is bounded and $|K|$ is bounded away from zero, then* $\inf R \leq 3 \leq \sup R$.

Remark 1.1. *By using similar arguments to the ones used in this paper, the authors already obtained a classification of complete minimal hypersurfaces with constant Gauss-Kronecker curvature in a four dimensional space form. The results will appear in a forthcoming paper.*

2 Preliminaries and Notations

Let M^3 be a 3-dimensional hypersurface in a unit sphere $\mathbb{S}^4$. We choose a local orthonormal frame field $\{e_1, \ldots, e_4\}$ in $\mathbb{S}^4$, so that, restricted to M^3, e_1, e_2, e_3 are tangent to M^3. Let $\{\omega_1, \ldots, \omega_4\}$ denote the dual co-frame field in $\mathbb{S}^4$. We use the following convention for the range of the indices: A, B, C, D range from 1 to 4 and i, j, k range from 1 to 3. The structure equations of $\mathbb{S}^4$ are given by

$$d\omega_A = -\sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$d\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2} \sum_{C,D} K_{ABCD}\, \omega_C \wedge \omega_D,$$

where $K_{ABCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}$ is the curvature tensor of $\mathbb{S}^4$. Since $\omega_4 = 0$

on M^3, by *Cartan's Lemma* we have

$$\omega_{4i} = \sum_j h_{ij}\omega_j, \; h_{ij} = h_{ji} \; . \tag{2.1}$$

We call $h = \sum_{i,j} h_{ij}\omega_i\omega_j$, the eigenvalues λ_i of the matrix (h_{ij}), $H = \sum_i h_{ii} = \sum_i \lambda_i$ and $K = \det(h_{ij}) = \prod_i \lambda_i$, respectively, the *second fundamental form*, the *principal curvatures*, the *mean curvature* and the *Gauss-Kronecker curvature* of M^3.

The structure equations of M^3 are given by

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$
$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum_{k,\ell} R_{ijk\ell}\omega_k \wedge \omega_\ell.$$

Using the formulas above we obtain the Gauss equation

$$R_{ijk\ell} = K_{ijk\ell} + h_{ik}h_{j\ell} - h_{i\ell}h_{jk}. \tag{2.2}$$

We recall that M^3 is a *minimal* hypersurface if its mean curvature is identically zero. From now on, we assume that M^3 is minimal. In this situation, its Ricci curvature tensor and scalar curvature are given, respectively, by

$$R_{ij} = 2\delta_{ij} - \sum_k h_{ik}h_{jk}, \tag{2.3}$$
$$R = 6 - S, \quad \text{where} \quad S = \sum_{i,j} h_{ij}^2 \text{ is the squared norm of } h. \tag{2.4}$$

It follows from (2.4) that R is constant if and only if S is constant.

The covariant derivative ∇h of the second fundamental form h of M^3 with components h_{ijk} is given by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{jk}\omega_{ik} + \sum_k h_{ik}\omega_{jk}.$$

Then the exterior derivative of (2.1) together with the structure equations yields the Codazzi equation

$$h_{ijk} = h_{ikj} = h_{jik}. \tag{2.5}$$

Hence h_{ijk} is symmetric on the indices i, j, k.

Similarly, we have the second covariant derivative $\nabla^2 h$ of h with components $h_{ijk\ell}$ as follows

$$\sum_\ell h_{ijk\ell}\omega_\ell = dh_{ijk} + \sum_\ell h_{\ell jk}\omega_{i\ell} + \sum_\ell h_{i\ell k}\omega_{j\ell} + \sum_\ell h_{ij\ell}\omega_{k\ell}.$$

For any fixed point p on M^3, we can choose a local orthonormal frame field $\{e_1, e_2, e_3\}$ such that

$$h_{ij} = \lambda_i \delta_{ij}.$$

The following formulas can be found in Peng and Terng [5].

$$h_{ijij} - h_{jiji} = (\lambda_i - \lambda_j)(1 + \lambda_i\lambda_j). \tag{2.6}$$

$$\Delta h_{ij} = (3 - S)h_{ij}. \tag{2.7}$$

$$\frac{1}{2}\Delta S = \sum_{i,j,k} h_{ijk}^2 + (3 - S)S. \tag{2.8}$$

The proof of our results relies heavily on the well known *Generalized Maximum Principle* due to H. Omori [4].

Lemma 2.1. *Let M^n be an n-dimensional complete Riemannian manifold whose sectional curvature is bounded from below and $f : M^n \to \mathbb{R}$ be a smooth function which is bounded from above on M^n. Then there is a sequence of points $\{p_k\}$ in M^n such that*

$$\lim_{k\to\infty} f(p_k) = \sup f;\ \lim_{k\to\infty} |\nabla f(p_k)| = 0 \text{ and}$$

$$\limsup_{k\to\infty} max\{(Hess_f(p_k))(X, X) : |X| = 1\} \leq 0,$$

where $Hess_f$ denotes the Hessian of f.

3 Proof of Theorem 1.1

The inequality $\sup S \geq 3$ is a particular case of the one stablished by Cheng in [1]. For reader's convenience we shall prove it here. Let us assume on the contrary that $\sup S < 3$. As S is bounded from (2.2) we see that the sectional curvatures are bounded from below. So, by using Lemma 2.1 we obtain a sequence $\{p_k\}$ of points in M^3 such that

$$\begin{gathered}\lim_{k\to\infty} S(p_k) = \sup S;\ \lim_{k\to\infty} |\nabla S(p_k)| = 0 \\ \text{and } \limsup_{k\to\infty}(S_{ii}(p_k)) \leq 0.\end{gathered} \tag{3.1}$$

By evaluating (2.8) at p_k and taking the limit for $k \to \infty$, from (3.1) we arrive to

$$\sup S(3 - \sup S) \leq \limsup_{k\to\infty} \frac{1}{2}\Delta S(p_k) \leq \frac{1}{2}\sum_i \limsup_{k\to\infty} S_{ii}(p_k) \leq 0. \tag{3.2}$$

This implies that $\sup S = 0$, i.e, M^3 is totally geodesic which contradicts our hypothesis that $|K|$ is bounded away from zero. Hence, we have $\sup S \geq 3$.

Now let us prove the inequality $\inf S \leq 3$. As K does not vanish, the function $F = \log|\det(h_{ij})|$ is globally defined on M^3 and is smooth. For any fixed point $p \in M^3$ we can take a local orthonormal frame field $\{e_1, e_2, e_3\}$ such that $h_{ij} = \lambda_i \delta_{ij}$ at p. According to Peng-Terng (see [5] pp 15) the Laplacian of F is given by

$$\Delta F = -\sum_{ijk} \frac{1}{\lambda_i \lambda_j} h_{ijk}^2 + \sum_{ik} \frac{1}{\lambda_i} h_{iikk}. \tag{3.3}$$

Since M^3 is minimal, we have $\sum_k h_{kkii} = H_{ii} = 0$, for all i. Together with (2.6) this gives

$$\begin{aligned}&\sum_{ik} \frac{1}{\lambda_i} h_{iikk} = \sum_{ik} \frac{1}{\lambda_i}\left[h_{kkii} + (\lambda_i - \lambda_k)(1 + \lambda_i\lambda_k)\right] = \\ &\sum_{ik} \frac{1}{\lambda_i}(\lambda_i - \lambda_k)(1 + \lambda_i\lambda_k) = 3(3 - S) = -3(S - 3).\end{aligned} \tag{3.4}$$

Notice that *Codazzi equation* (2.5) yields

$$\frac{1}{\lambda_i\lambda_j} h_{ijk}^2 = \frac{1}{\lambda_j\lambda_i} h_{jik}^2.$$

Then the coefficient of h_{123}^2 in $\sum_{ijk} \frac{1}{\lambda_i\lambda_j} h_{ijk}^2$ can be given by

$$2\left(\frac{1}{\lambda_1\lambda_2}+\frac{1}{\lambda_1\lambda_3}+\frac{1}{\lambda_2\lambda_3}\right)=\frac{2H}{K}=0$$

and we may write

$$\sum_{ijk} \frac{1}{\lambda_i\lambda_j} h_{ijk}^2 = \sum_i \sum_{j\neq i, k\neq i, j<k}\left[\frac{1}{\lambda_i^2}h_{iii}^2+\left(\frac{1}{\lambda_j^2}+\frac{2}{\lambda_i\lambda_j}\right)h_{jji}^2+\left(\frac{1}{\lambda_k^2}+\frac{2}{\lambda_i\lambda_k}\right)h_{kki}^2\right]. \tag{3.5}$$

Let i, j, k be pairwise distinct indices. Bearing in mind that M^3 is minimal, we have $\lambda_i+\lambda_j=-\lambda_k$, $\lambda_i+\lambda_k=-\lambda_j$ and $h_{iii}=-(h_{jji}+h_{kki})$ which implies

$$\begin{aligned}
&\frac{1}{\lambda_i^2}h_{iii}^2+\left(\frac{1}{\lambda_j^2}+\frac{2}{\lambda_i\lambda_j}\right)h_{jji}^2+\left(\frac{1}{\lambda_k^2}+\frac{2}{\lambda_i\lambda_k}\right)h_{kki}^2= \\
&\frac{1}{\lambda_i^2}(h_{jji}+h_{kki})^2+\left(\frac{1}{\lambda_j^2}+\frac{2}{\lambda_i\lambda_j}\right)h_{jji}^2+\left(\frac{1}{\lambda_k^2}+\frac{2}{\lambda_i\lambda_k}\right)h_{kki}^2= \\
&\left(\frac{1}{\lambda_i^2}+\frac{2}{\lambda_i\lambda_j}+\frac{1}{\lambda_j^2}\right)h_{jji}^2+\left(\frac{1}{\lambda_i^2}+\frac{2}{\lambda_i\lambda_k}+\frac{1}{\lambda_k^2}\right)h_{kki}^2+ \\
&\frac{2}{\lambda_i^2}h_{jji}h_{kki}=\left(\frac{1}{\lambda_i}+\frac{1}{\lambda_j}\right)^2h_{jji}^2+\left(\frac{1}{\lambda_i}+\frac{1}{\lambda_k}\right)^2h_{kki}^2+\frac{2}{\lambda_i^2}h_{jji}h_{kki}= \\
&\frac{\lambda_k^2}{\lambda_i^2\lambda_j^2}h_{jji}^2+\frac{\lambda_j^2}{\lambda_i^2\lambda_k^2}h_{kki}^2+\frac{2}{\lambda_i^2}h_{jji}h_{kki}=\frac{1}{K^2}\left(\lambda_k^2h_{jji}+\lambda_j^2h_{kki}\right)^2.
\end{aligned} \tag{3.6}$$

Inserting (3.6) into (3.5) we obtain

$$\sum_{ijk} \frac{1}{\lambda_i\lambda_j} h_{ijk}^2 = \frac{1}{K^2}\left[\left(\lambda_3^2h_{221}+\lambda_2^2h_{331}\right)^2+\left(\lambda_3^2h_{112}+\lambda_1^2h_{332}\right)^2+\left(\lambda_2^2h_{113}+\lambda_1^2h_{223}\right)^2\right]. \tag{3.7}$$

It follows from (3.3), (3.4) and (3.7) that

$$\Delta F = -\frac{1}{K^2}\Big[\big(\lambda_3^2 h_{221} + \lambda_2^2 h_{331}\big)^2 + \big(\lambda_3^2 h_{112} + \lambda_1^2 h_{332}\big)^2 + \big(\lambda_2^2 h_{113} + \lambda_1^2 h_{223}\big)^2\Big] - 3(S-3). \tag{3.8}$$

As S is bounded, we have already seen that the sectional curvatures of M^3 are bounded from below. Further, since $|K|$ is bounded away from zero, $F = \log|\det(h_{ij})|$ is bounded from below, so we may apply the *Generalized Maximum Principle due to Omori* to the function F to obtain a sequence $\{p_k\}$ of points in M^3 such that

$$\lim_{k\to\infty} F(p_k) = \inf F;\ \lim_{k\to\infty} |\nabla F(p_k)| = 0 \quad \text{and} \liminf_{k\to\infty}(F_{ii}(p_k)) \geq 0. \tag{3.9}$$

In view of (3.8) we get the inequality

$$\Delta F \leq -3(S-3). \tag{3.10}$$

Evaluating (3.10) at $\{p_k\}$ and making $k \to \infty$, from (3.9) we obtain

$$0 \leq \sum_i \liminf_{k\to\infty} F_{ii}(p_k) \leq \Delta F \leq \liminf_{k\to\infty} 3(3 - S(p_k)). \tag{3.11}$$

From (3.11) we deduce that $\inf S \leq 3$, which completes our proof.

□

Remark 3.1. *We would like to emphasize that the hypothesis that $|K|$ is bounded away from zero cannot be dropped, as shows the following example.*

Example 3.1. *The hypersurface M^3 in $\mathbb{S}^4$ defined by the equation*

$$2x_5^3 + 3(x_1^2 + x_2^2)x_5 - 6(x_3^2 + x_4^2)x_5 + 3\sqrt{3}(x_1^2 - x_2^2)x_4 + 6\sqrt{3}x_1x_2x_3 = 2$$

was investigated by E. Cartan, who proved that this space is a homogeneous Riemannian manifold $SO(3)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ and that its principal curvatures are $-\sqrt{3}, 0, \sqrt{3}$. Therefore, M^3 has $\inf S = S = 6$.

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