*Matemática Contemporânea, Vol 34, 135-154*


# FOLIATIONS BY $(n-1)$-UMBILICAL SPACELIKE HYPERSURFACES

**A. G. Colares** * **O. Palmas**†

*Dedicated to Professor Manfredo do Carmo on the occasion of his 80$^{th}$ birthday*

## Introduction

In the study of spacelike hypersurfaces in Lorentzian manifolds appears as an important mark Goddard's conjecture, posed in 1977 [7]: the only complete constant mean curvature spacelike hypersurfaces (i. e., the induced metric is Riemannian) in the de Sitter space $\mathbb{S}_1^{n+1}$ are the umbilical ones. The first answer to Goddard's conjecture was given by Dajczer and Nomizu in 1981 [4], when they exhibited an example of a (flat) complete surface in the 3-dimensional de Sitter space $\mathbb{S}_1^3$ with constant mean curvature which is not umbilical. A fundamental answer to the mentioned conjecture was given by Montiel in 1988 [9]. He proved that the only compact constant mean curvature spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ are the umbilical ones. Moreover, Montiel described all of them. Since then, the interest on the subject increased, at least in two directions. The first one is the search of conditions in more general Lorentzian manifolds to guarantee that constant mean curvature spacelike hypersurfaces are umbilical. In 1999 Montiel [11] proved that in a Lorentzian manifold a closed conformal vector field determines a foliation by constant mean curvature umbilical hypersurfaces and he was able to show, with additional hypotheses, that in such a space every constant mean curvature compact spacelike hypersurface is umbilical. The second direction points to the construction of examples of non-umbilical complete constant mean curvature spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$,

*Partially supported by DGAPA–UNAM, México, under Project IN118508 and PRONEX, Brazil.
†Partially supported by DGAPA–UNAM, México, under Project IN118508 and PRONEX, Brazil.

initiated by Dajczer and Nomizu and followed by Akutagawa [1], Ramanathan [14] and Ki, Kim and Nakagawa [8].

Montiel also constructed examples of non-umbilical complete constant mean curvature spacelike hypersurfaces in the de Sitter space including the hyperbolic cylinders ([9] and [10]), so called because they can be generated by hyperbolas. We observe that all the mentioned non-umbilical examples are $(n-1)$-umbilical, that is, they are not umbilical because at every point $(n-1)$ principal curvatures are equal and one is distinct from the others (see Definition 1.1). More precisely, let $\bar{M}_1^{n+1}$ be a Lorentzian manifold with semi-Riemannian connection $\bar{\nabla}$ and $M$ an orientable spacelike hypersurface with a unit timelike normal vector field $N$. We say that $M$ is $(n-1)$-umbilical if there is a $(n-1)$-dimensional distribution $\mathcal{D} \subset TM$, as well as a $C^\infty$ function $\phi$ such that

$$\mathcal{D} = \{ \ X \in TM \mid \bar{\nabla}_X N = \phi X \ \}. \tag{1}$$

Here we construct a family of new examples of $(n-1)$-umbilical spacelike hypersurfaces of constant mean curvature in $\mathbb{S}_1^{n+1}$. More precisely, we prove (see Theorem 2.5):

*For every real number $H$ there is a 1-parameter family of $(n-1)$-umbilical spacelike hypersurfaces $M^n \subset \mathbb{S}_1^{n+1}$ with constant mean curvature equal to $H$. If $H = 1$ or $H = 2\sqrt{n-1}/n$, this family contains one cylinder; that is, a non-totally umbilical hypersurface with constant principal curvatures. If $H > 2\sqrt{n-1}/n$ and $H \neq 1$, this family contains two different cylinders.*

*Moreover, for $H > 2\sqrt{n-1}/n$ there is a subfamily of such $(n-1)$-umbilical spacelike hypersurfaces which are complete.*

Consideration of the above examples of $(n-1)$-umbilical spacelike hypersurfaces with constant mean curvature leads to the following question:

*Under which conditions a given Lorentzian manifold can be foliated by $(n-1)$-umbilical spacelike hypersurfaces of constant mean curvature?*

In Section 3 we introduce the notion of a timelike closed partially conformal vector field on a Lorentzian manifold $\bar{M}_1^{n+1}$ (see Definition 3.1) and prove that it is the right tool to solve our question. Let $\bar{M}_1^{n+1}$ be a Lorentzian manifold with semi-Riemannian connection $\bar{\nabla}$ and $K \in \mathfrak{X}(\bar{M})$ a timelike vector field. We say that $K$ is *closed partially conformal* in $\bar{M}$ if there is a unit vector field $W \in \mathfrak{X}(\bar{M})$ everywhere orthogonal to $K$ such that

$$\bar{\nabla}_X K = \phi X \text{ for } \langle X, W\rangle = 0 \quad \text{and} \quad \bar{\nabla}_W K = \psi W \tag{2}$$

for some functions $\phi, \psi : \bar{M} \to \mathbb{R}$. It is said that $W$ is *associated* to $K$.

We note that the transformations corresponding to the flow of our partially conformal vector field are partially conformal transformations as defined by Tanno (see [15] and [16]).

Then we prove the following fact (see Theorem 3.2 and Lemma 3.3):

*If $\bar{M}_1^{n+1}$ is a Lorentzian manifold endowed with a timelike closed partially conformal vector field $K$, then the distribution $K^{\perp}$ is involutive and each leaf of the corresponding foliation is a $(n-1)$-umbilical spacelike hypersurface with $n-1$ equal and constant principal curvatures.*

Here $K^{\perp}$ denotes the distribution defined by taking the orthogonal complement of $K$ at each point.

We close this paper with Section 4, where we present examples of timelike closed partially conformal vector fields and their corresponding $(n-1)$-umbilical foliations of open subsets of Lorentzian space forms.

# 1 Preliminaries

We will denote by $\bar{M}_\nu^{n+1}$, or simply by $\bar{M}$, a $(n+1)$-dimensional semi-Riemannian manifold, endowed with a metric tensor $\langle\, , \rangle$ of index $\nu \geq 0$. In particular, if $\nu = 0$, $\bar{M}$ is Riemannian, while if $\nu = 1$, $\bar{M}$ is Lorentzian. Also, $\bar{\nabla}$ will denote the semi-Riemannian connection of $\bar{M}$.

For example, let $\mathbb{R}_\nu^{n+1}$ be the $(n+1)$-dimensional vector space with metric tensor

$$\langle v, w \rangle = -\sum_{i=1}^{\nu} v_i w_i + \sum_{j=\nu+1}^{n+1} v_j w_j,$$

where $v = (v_1, \ldots, v_{n+1})$ and $w = (w_1, \ldots, w_{n+1})$.

As another example, for $n \geq 1$ and $c > 0$ we define

$$\mathbb{S}_\nu^{n+1}(c) = \left\{ \; p \in \mathbb{R}_\nu^{n+2} \; \middle| \; \langle p, p \rangle = \frac{1}{c} \; \right\}.$$

This is a space with constant (positive) curvature $c$. We call $\mathbb{R}_\nu^{n+2}$ the *ambient space* of $\mathbb{S}_\nu^{n+1}(c)$. If $c = 1$, we simply denote this space as $\mathbb{S}_\nu^{n+1}$.

To standarize our notation, we set $\mathbb{R}_\nu^{n+2}$ as the ambient space of $\mathbb{R}_\nu^{n+1}$, that is,

$$\mathbb{R}_\nu^{n+1} = \{ \; (x_1, \ldots, x_{n+2}) \in \mathbb{R}_\nu^{n+2} \mid x_{n+2} = 0 \; \}.$$

We will denote by $\mathbb{Q}_1^{n+1}(c)$ the standard $(n+1)$-dimensional Lorentzian manifold of constant curvature $c \geq 0$; that is, for $c = 0$ we have the *Lorentz-Minkowski space* $\mathbb{R}_1^{n+1}$ and for $c > 0$ we get the *de Sitter space* $\mathbb{S}_1^{n+1}(c)$.

Given a semi-Riemannian manifold $\bar{M}$, a submanifold $M \subseteq \bar{M}$ is *spacelike* if the metric induced on $M$ is Riemannian. It is clear that if $M$ is a spacelike *hypersurface* (that is, it has codimension 1), then $\bar{M}$ has to be Riemannian or Lorentzian.

We define now the class of hypersurfaces we are interested in.

**Definition 1.1.** *Let $\bar{M}_\nu^{n+1}$ be a semi-Riemannian manifold and $M$ an orientable spacelike hypersurface $M \subseteq \bar{M}$; that is, there is a unit timelike vector field $N$ everywhere orthogonal to $M$. We say that $M$ is* $k$-umbilical *if there is a $k$-dimensional distribution $\mathcal{D} \subset TM$, as well as a $C^\infty$ function $\phi$ such that*

$$\mathcal{D} = \{ \; X \in TM \mid \bar{\nabla}_X N = \phi X \; \}. \tag{3}$$

It turns out that a $k$-umbilical hypersurface has $k$ equal principal curvatures. (See, for example, [3].)

## 2 Examples

We recall briefly the known examples of $(n-1)$-umbilical spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$. The first one was given in 1981 by Dajczer and Nomizu [4] in the 3-dimensional case. For $r>0$, let $f:\mathbb{R}^2\to\mathbb{S}_1^3$, $f=f(x,y)$ be given by

$$\left(r\cosh\frac{x}{r}, r\sinh\frac{x}{r}, \sqrt{1+r^2}\cos\frac{y}{\sqrt{1+r^2}}, \sqrt{1+r^2}\sin\frac{y}{\sqrt{1+r^2}}\right). \tag{4}$$

It is proved that $f$ is a flat immersion, with principal curvatures given by

$$\frac{r}{\sqrt{1+r^2}} \quad \text{and} \quad \frac{\sqrt{1+r^2}}{r},$$

which clearly are distinct, so the image of $f$ is a 1-umbilical (hyper)surface.

In [9], Montiel constructed more examples in $\mathbb{S}_1^{n+1}$, as follows: Take $\rho>0$, $1\le k\le n-1$ and consider

$$M=\{\,(x_1,\dots,x_{n+2})\in\mathbb{S}_1^{n+1}\mid -x_1^2+x_2^2+\cdots+x_{k+1}^2=-\sinh^2\rho\,\}. \tag{5}$$

This is a spacelike hypersurface in $\mathbb{S}_1^{n+1}$ isometric to the Riemannian product

$$\mathbb{H}^k(1-\coth^2\rho)\times\mathbb{S}^{n-k}(1-\tanh^2\rho)$$

of a $k$-dimensional hyperbolic space and a $(n-k)$-dimensional sphere of constant sectional curvatures $(1-\coth^2\rho)$ and $(1-\tanh^2\rho)$, respectively. $M$ has $k$ principal curvatures equal to $\coth\rho$ and $(n-k)$ principal curvatures equal to $\tanh\rho$; thus, $M$ is $k$-umbilical, as well as $(n-k)$-umbilical. If $k=1$, $M$ is called a *hyperbolic cylinder*. Note also that if $n=2$ and $k=1$ we recover the Dajczer-Nomizu example, with $r=\sinh\rho$.

In 1991, Ki, Kim and Nakagawa [8] studied the spacelike hypersurfaces in $\mathbb{Q}_1^{n+1}(c)$ and gave examples isometric to the products of Riemannian space forms. In the case of the de Sitter space $\mathbb{S}_1^{n+1}$, they considered the family of spacelike hypersurfaces $\mathbb{H}^k(c_1)\times\mathbb{S}^{n-k}(c_2)$ given by

$$\left\{(x,y)\in\mathbb{S}_1^{n+1}\subset\mathbb{R}_1^{n+2}=\mathbb{R}_1^{k+1}\times\mathbb{R}^{n-k+1}\;\middle|\;\langle x,x\rangle=\frac{1}{c_1},\ \langle y,y\rangle=\frac{1}{c_2}\right\},$$

where $c_1 < 0$, $c_2 > 0$, $1/c_1 + 1/c_2 = 1$ and $k = 1, \dots, n-1$. The principal curvatures are $\sqrt{1-c_1}$ with multiplicity $k$ and $\sqrt{1-c_2}$ with multiplicity $n-k$. Thus, for $k=1$ we recover Montiel's hyperbolic cylinders, while for $k = n-1$ we obtain the hypersurfaces $\mathbb{H}^{n-1}(c_1) \times \mathbb{S}^1(c_2)$.

Note that the examples cited have constant principal curvatures. In the sequel we will call a $(n-1)$-umbilical (non-totally umbilical) hypersurface with constant principal curvatures a *cylinder*. We shall prove here that these cylinders in $\mathbb{S}_1^{n+1}$ belong in fact to a whole family of $(n-1)$-umbilical spacelike hypersurfaces; namely, to a family of *rotation hypersurfaces*.

A general definition of rotation hypersurfaces was given in the Riemannian case by do Carmo and Dajczer in [5], definition which was extended later to some Lorentzian manifolds (see for example [12]). For the sake of completeness, we give the definition of these rotation hypersufaces in $\mathbb{Q}_1^{n+1}(c)$.

Recall from the Preliminaries section that each $\mathbb{Q}_1^{n+1}(c)$ has an ambient space of the form $\mathbb{R}_\nu^{n+2}$, $\nu = 1, 2$. We say that an *orthogonal transformation* of $\mathbb{R}_\nu^{n+2}$ is a metric-preserving linear map. By restriction, these orthogonal transformations induce all isometries of $\mathbb{Q}_1^{n+1}(c)$.

Let $P^k$ be a $k$-dimensional vector subspace of $\mathbb{R}_\nu^{n+2}$. $O(P^k)$ will denote the set of orthogonal transformations of $\mathbb{R}_\nu^{n+2}$ with positive determinant that leave $P^k$ pointwise fixed.

Fix a 3-dimensional space $P^3$, a subspace $P^2 \subset P^3$, and a regular, spacelike curve $C$ in $\mathbb{Q}_1^{n+1}(c) \cap (P^3 - P^2)$, parametrized by arc length. The orbit of $C$ under $O(P^2)$ is called the *rotation spacelike hypersurface $M$ in $\mathbb{Q}_1^{n+1}(c)$ generated by $C$*. $M$ is *spherical* (*hyperbolic, parabolic*, resp.) whenever the ambient metric restricted to $P^2$ is a Lorentzian (Riemannian, degenerate, resp.) metric.

After giving the general definition of a rotation hypersurface in the Riemannian context, do Carmo and Dajczer imposed the condition of having constant mean curvature, studying and classifying these hypersurfaces. Following similar methods, spherical rotation hypersurfaces in $\mathbb{S}_1^{n+1}$ with constant mean curvature were described by the authors in [2]. Thus, in this paper we will describe in detail the hyperbolic rotation hypersurfaces in $\mathbb{S}_1^{n+1}$ and make some comments about the spherical cases. We will consider here the parabolic rotation case only in Remark 2.3.

Let $\{e_1, e_2, \dots, e_{n+2}\}$ be the canonical basis of $\mathbb{R}_1^{n+2}$, so that

$$\langle e_1, e_1 \rangle = -1 \quad \text{and} \quad \langle e_i, e_i \rangle = 1 \text{ for } i > 1.$$

Also, let $P^2 = \text{span}(e_{n+1}, e_{n+2})$ and $P^3 = \text{span}(e_1, e_{n+1}, e_{n+2})$. The profile curve generating the rotation hypersurface is given by

$$(x_1(s), 0, \dots, 0, x_{n+1}(s), x_{n+2}(s)),$$

where

$$-x_1^2 + x_{n+1}^2 + x_{n+2}^2 = 1 \qquad \text{and} \qquad -\dot{x}_1^2 + \dot{x}_{n+1}^2 + \dot{x}_{n+2}^2 = 1.$$

Here the dots denote derivative with respect to $s$.

Now take $\Phi(t_1, \dots, t_{n-1}) = (\varphi_1, \dots, \varphi_n)$ as an orthogonal parametrization of the unit hyperbolic space $\mathbb{H}^{n-1} \subset \mathbb{R}_1^n$, so that

$$-\varphi_1^2 + \varphi_2^2 + \dots + \varphi_n^2 = -1, \qquad \varphi_1 > 0.$$

Thus,

$$f(t_1, \dots, t_{n-1}, s) = (x_1(s)\Phi(t_1, \dots, t_{n-1}), x_{n+1}(s), x_{n+2}(s)) \tag{6}$$

is the desired parametrization of the spacelike hyperbolic rotation hypersurface generated by the curve $(x_1(s), 0, \dots, 0, x_{n+1}(s), x_{n+2}(s))$.

Differentiating equation (6), we have

$$E_i = \frac{\partial f}{\partial t_i} = \left( x_1 \frac{\partial \Phi}{\partial t_i}, 0, 0 \right), \ i = 1, \dots, n-1,$$

and

$$E_n = \frac{\partial f}{\partial s} = (\dot{x}_1 \Phi, \dot{x}_{n+1}, \dot{x}_{n+2})$$

so that

$$\langle E_i, E_j \rangle = x_1^2 \left\langle \frac{\partial \Phi}{\partial t_i}, \frac{\partial \Phi}{\partial t_j} \right\rangle \text{ for } i,j = 1,\dots,n-1,$$

while

$$\langle E_n, E_n \rangle = 1 \quad \text{and} \quad \langle E_i, E_n \rangle = 0 \text{ for } i = 1,\dots,n-1.$$

We choose the timelike unit normal vector $N$ as

$$(-(\dot{x}_{n+1}x_{n+2} - x_{n+1}\dot{x}_{n+2})\Phi, (x_1\dot{x}_{n+2} - \dot{x}_1 x_{n+2}), (\dot{x}_1 x_{n+1} - x_1\dot{x}_{n+1})).$$

Now it is easy to prove that

$$\bar{\nabla}_{E_i} N = -\frac{\dot{x}_{n+1}x_{n+2} - x_{n+1}\dot{x}_{n+2}}{x_1} E_i, \text{ for } i = 1,\dots,n-1. \tag{7}$$

This fact tells us that the coordinate curves are lines of curvature and that the principal curvatures along the $t_i$–curves are

$$\kappa_i = -\frac{\dot{x}_{n+1}x_{n+2} - x_{n+1}\dot{x}_{n+2}}{x_1}. \tag{8}$$

We use $-x_1^2 + x_{n+1}^2 + x_{n+2}^2 = 1$ to write this in terms of $x = x_1$ alone. Let

$$x_{n+1} = \sqrt{1+x^2}\cos\theta \quad \text{and} \quad x_{n+2} = \sqrt{1+x^2}\sin\theta, \tag{9}$$

for an unknown function $\theta$, which may be obtained deriving the above expressions and using $-\dot{x}^2 + \dot{x}_{n+1}^2 + \dot{x}_{n+2}^2 = 1$. We have

$$\dot{\theta}^2 = \frac{x^2 + \dot{x}^2 + 1}{x^2 + 1}.$$

We differentiate $x_{n+1}$ and $x_{n+2}$ in (9), use the above expression for $\dot{\theta}$ and (8) to express the principal curvatures $\kappa_i$ in terms of $x$ as

$$\kappa_i = \frac{\sqrt{x^2 + \dot{x}^2 + 1}}{x}.$$

The expression for $\kappa_n$ is obtained in a similar but longer way, differentiating $\dot{x}_{n+1}$ and $\dot{x}_{n+2}$, using (8), (9) and the expression for $\dot{\theta}$. In the following Proposition we summarize this analysis; see also [12].

**Proposition 2.1.** *The hyperbolic rotation hypersurface in $\mathbb{S}_1^{n+1}$ parametrized by (6) is $(n-1)$-umbilical; moreover, it has principal curvatures given by*

$$\kappa_i = \frac{\sqrt{x^2+\dot{x}^2+1}}{x} \quad \text{and} \quad \kappa_n = \frac{\ddot{x}+x}{\sqrt{x^2+\dot{x}^2+1}}, \tag{10}$$

*where $i = 1, \ldots, n-1$ and $\delta = 1, -1, 0$.*

Using (10), we get that the mean curvature $H$ of $M$ is

$$nH = (n-1)\frac{\sqrt{x^2+\dot{x}^2+1}}{x} + \frac{\ddot{x}+x}{\sqrt{x^2+\dot{x}^2+1}}. \tag{11}$$

If we suppose that $H$ is constant, this equation has a first integral, namely,

$$G(x,\dot{x}) = x^{n-1}\left(\sqrt{x^2+\dot{x}^2+1} - Hx\right). \tag{12}$$

We use $G$ to obtain a classification of the spacelike hyperbolic rotation hypersurfaces with constant mean curvature in $\mathbb{S}_1^{n+1}$.

Of particular importance here are the critical points of $G$ of the form $(x,0)$, which appear whenever $\partial G/\partial x$ and $\partial G/\partial \dot{x}$ vanish. These conditions give the equation

$$x^2 - nHx\sqrt{x^2+1} + (n-1)(x^2+1) = 0.$$

To solve it, we make the substitution $x = \sinh\rho$ and divide by $\cosh^2\rho$ to get

$$\tanh^2\rho - nH\tanh\rho + (n-1) = 0.$$

Then

$$\tanh\rho = \frac{nH \pm \sqrt{n^2H^2 - 4(n-1)}}{2}.$$

As $|\tanh\rho| < 1$, it may be seen easily that we have the restriction $|H| > 1$ and that we must choose the minus sign in the expression above; that is,

$$\tanh\rho = \frac{nH - \sqrt{n^2H^2 - 4(n-1)}}{2}. \tag{13}$$

As a consequence, the function $G$ exactly has one critical point $(x,0)$ whenever $|H| > 1$. Each critical point corresponds to a spacelike hyperbolic rotation hypersurface in $\mathbb{S}_1^{n+1}$ with constant principal curvatures.

**Remark 2.2.** *Dajczer and Nomizu's example given by (4) is a hyperbolic rotation hypersurface. In (6), set* $x_1(s) = r$, $\Phi(t) = (\cosh t/r, \sinh t/r)$ *and*

$$x_{n+1}(s) = \sqrt{1+r^2}\cos\frac{s}{\sqrt{1+r^2}} \quad \text{and} \quad x_{n+2}(s) = \sqrt{1+r^2}\sin\frac{s}{\sqrt{1+r^2}}.$$

*In fact, this example may be viewed* also *as an spherical rotation hypersurface. We will not give the details here, since the parametrization of such rotation hypersurfaces is quite similar to (6).*

In [2], the authors jointly with A. Brasil Jr. analyzed the spherical rotation hypersurfaces and showed the existence of two different cylinders with constant mean curvature if $2\sqrt{n-1}/n \leq H < 1$ and the existence of one such cylinder if $H \geq 1$. As pointed out after equation (13), for each $H > 1$ there is also another cylinder given as a hyperbolic rotation hypersurface.

**Remark 2.3.** *It may be observed that there is a "missing" cylinder in the case* $H = 1$ *and one may be tempted to look for it by analyzing the parabolic rotation case. Let us show briefly what happens in this situation. It can be proved that the principal curvatures of a parabolic rotation hypersurface satisfy*

$$\kappa_i = \frac{\sqrt{x^2+\dot{x}^2}}{x} \quad \text{and} \quad \kappa_n = \frac{\ddot{x}+x}{\sqrt{x^2+\dot{x}^2}}. \tag{14}$$

*If we make an analysis similar to that of the hyperbolic rotation case, obtaining a function* $G$ *analogous to that of equation (12), we calculate*

$$\frac{\partial G}{\partial x}(x,0) = (n+1)(1-H)x^{n-1};$$

*this fact means that, for* $H = 1$, *every point* $(x,0)$ *is a critical point of* $G$. *But then all principal curvatures in (14) are equal and the corresponding hypersurface is totally umbilical.*

We continue with our analysis of hyperbolic rotation case, studying the level curves of $G$ near a critical point $(x,0)$, with $x = \sinh\rho$ and $\rho$ satisfying (13). An elementary analysis shows that this critical point is a saddle point. A typical configuration is depicted in Figure 1.

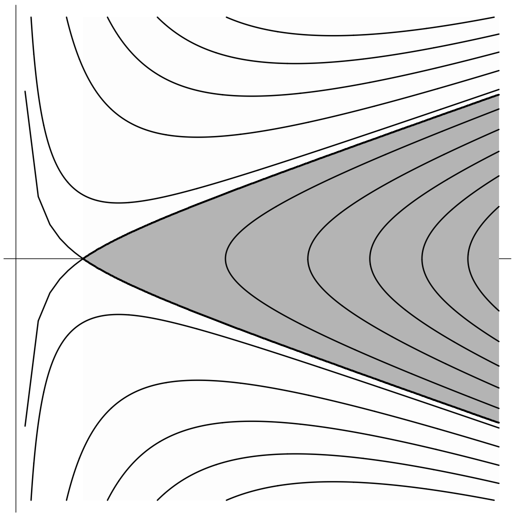

Figure 1: Level curves of the function $G$ defined by equation (12). The level curves ending at the saddle point determine four subregions; the shaded one is the region I referenced to in the text.

The level curves ending at the saddle point divide the region $x > 0$ in four subregions, from which we will analyze the subregion situated to the right of the saddle point, which we will call region I.

**Lemma 2.4.** *The level curves contained in region I give rise to a family of* complete *hyperbolic rotation hypersurfaces.*

**Proof:** Let us rewrite the original mean curvature equation (11) as the system

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= nH\sqrt{u^2+v^2+1} - (n-1)\frac{u^2+v^2+1}{u} - u, \end{aligned}$$

thus defining a vector field $X = (X_1, X_2)$ in the $(u, v)$-plane (or, the $(x, \dot{x})$-plane). We have to prove that the trajectories of $X$ passing through points of the region I are defined for all $s \in \mathbb{R}$. By a result in [6], we may prove equivalently that there is a differentiable proper function $h$ such that $|Xh|$ is uniformly bounded in the region I. We define

$$h(u,v) = \log\left(u^2 + v^2 + 1\right),$$

which is clearly a differentiable proper function. We will estimate

$$|Xh| = \left| X_1 \frac{\partial h}{\partial u} + X_2 \frac{\partial h}{\partial v} \right|.$$

After a few calculations, we have

$$\begin{aligned} X_1 \frac{\partial h}{\partial u} + X_2 \frac{\partial h}{\partial v} &= \left( nH\sqrt{u^2+v^2+1} - (n-1)\frac{u^2+v^2+1}{u} \right) \frac{2v}{u^2+v^2+1} \\ &= \frac{2nHv}{\sqrt{u^2+v^2+1}} - 2(n-1)\frac{v}{u}. \end{aligned}$$

The first term in the last expression is uniformly bounded by, say, $2n|H|$. For the second term, we observe that for each constant $C$ the level curve $G(u,v) = C$ is a union of two curves $v = v(u)$ given by

$$v^2 = \left( \frac{C}{u^{n-1}} + Hu \right)^2 - u^2 - 1,$$

so that

$$\lim_{u\to\infty} \left(\frac{v}{u}\right)^2 = \lim_{u\to\infty} \left( \left( \frac{C}{u^n} + H \right)^2 - 1 - \frac{1}{u^2} \right) = H^2 - 1;$$

thus, the slopes of all these level curves tend to $\pm\sqrt{H^2-1}$ when $u$ goes to infinity, which implies that the slopes $v/u$ of the lines passing through the origin and the points of region I are uniformly bounded. Thus, the criterion given in [6] is satisfied and the trajectories of the vector field $X$ are complete, which in turn implies that the corresponding hyperbolic rotation hypersurfaces are complete.

The above analysis is completely similar in the spherical rotation case, thus we may obtain another family of complete hypersurfaces with constant mean curvature. We summarize our results as follows.

**Theorem 2.5.** *For every real number $H$ there is a 1-parameter family of $(n-1)$-umbilical spacelike hypersurfaces $M^n \subset \mathbb{S}_1^{n+1}$ with constant mean curvature* *equal to $H$. If $H = 1$ or $H = 2\sqrt{n-1}/n$, this family contains one cylinder; that is, a non-totally umbilical hypersurface with constant principal curvatures. If $H > 2\sqrt{n-1}/n$ and $H \neq 1$, this family contains two different cylinders.*

*Moreover, for $H > 2\sqrt{n-1}/n$ there is a subfamily of such $(n-1)$-umbilical spacelike hypersurfaces which are complete.*

To close this section, we state a useful characterization of $(n-1)$-umbilical hypersurfaces. The proof of this Theorem is entirely analogous to that given in [5] for the Riemannian case and we shall omit it.

**Theorem 2.6.** *Let $M^n$, $n \geq 3$, be a connected $(n-1)$-umbilical spacelike hypersurface in $\mathbb{Q}_1^{n+1}(c)$. Assume that the principal curvatures $\kappa_1, \dots, \kappa_n$ satisfy $\kappa_1 = \cdots = \kappa_{n-1} = \lambda \neq 0$ and $\kappa_n = \mu = \mu(\lambda)$, where $\lambda \neq \mu$. Then $M^n$ is contained in a rotation hypersurface.*

## 3 Closed partially conformal vector fields

The examples of the previous section show that the de Sitter space $\mathbb{S}_1^{n+1}$ has plenty of $(n-1)$-umbilical spacelike hypersurfaces with constant mean curvature. Moreover, the family of hyperbolic cylinders defined in (5), for $k = n-1$ and $\rho > 0$,

$$M = \{\, (x_1, \dots, x_{n+2}) \in \mathbb{S}_1^{n+1} \mid -x_1^2 + x_2^2 + \cdots + x_n^2 = -\sinh^2\rho \,\},$$

determine a foliation of the open set of $\mathbb{S}_1^{n+1}$ given by

$$\{\, (x_1, \dots, x_{n+2}) \in \mathbb{S}_1^{n+1} \mid -x_1^2 + x_2^2 + \cdots + x_n^2 < 0 \,\}.$$

This fact raises the question given in the Introduction, namely,

*Under which conditions a given Lorentzian manifold can be foliated by $(n-1)$-umbilical spacelike hypersurfaces of constant mean curvature?*

In our examples, the $(n-1)$-umbilical spacelike hypersurfaces satisfied a special condition given in equation (7). We add a similar condition in the following definition of some vector fields which will prove to be useful to answer our above question.

**Definition 3.1.** *Let $\bar{M}_1^{n+1}$ be a Lorentzian manifold with semi-Riemannian connection $\bar{\nabla}$. We say that a timelike vector field $K \in \mathfrak{X}(\bar{M})$ is* closed partially conformal *in $\bar{M}$ if there is a unit vector field $W \in \mathfrak{X}(\bar{M})$ everywhere orthogonal to $K$ and functions $\phi, \psi : \bar{M} \to \mathbb{R}$ such that*

$$\bar{\nabla}_X K = \phi X \text{ for } \langle X, W\rangle = 0 \quad \text{and} \quad \bar{\nabla}_W K = \psi W. \tag{15}$$

*In this context, $W$ is called the vector field* associated *to $K$.*

This notion is intimately related with that of closed conformal vector fields analyzed in detail in [11].

As $K$ is timelike, $|K| = \sqrt{-\langle K, K\rangle} \neq 0$, so we may define the unit vector field $N = K/|K|$. It is easy to see from Definition 3.1 that

$$\begin{aligned} &\bar{\nabla}_X N = \tfrac{\phi}{|K|} X \quad \text{ if } \langle X, W\rangle = \langle X, K\rangle = 0,\\ &\bar{\nabla}_W N = \tfrac{\psi}{|K|} W,\\ &\bar{\nabla}_N N = 0, \end{aligned} \tag{16}$$

so that $N$ defines a unit speed geodesic flow. Note that $\psi$ is related to the normal curvature $\kappa$ of the integral curves of $W$, since

$$\kappa = \langle \bar{\nabla}_W W, N\rangle = -\langle W, \bar{\nabla}_W N\rangle = -\frac{\psi}{|K|}.$$

Our next result justifies the introduction of closed partially conformal vector fields. Here and in the sequel $K^\perp$ denotes the distribution defined by taking the orthogonal complement of $K$ at each point; the distribution $W^\perp$ is defined in an analogous way.

**Theorem 3.2.** *Let $\bar{M}_1^{n+1}$ be a Lorentzian manifold possessing a closed partially conformal timelike vector field $K$. Then the distribution $K^\perp$ is involutive and each leaf of the foliation determined by $K^\perp$ is a $(n-1)$-umbilical hypersurface, thus having $n-1$ equal principal curvatures.*

**Proof:** First we will prove that $K^\perp$ is an involutive distribution. Let $X, Y$ be vector fields in $K^\perp$. Then

$$\langle [X,Y], K\rangle = \langle \bar\nabla_X Y - \bar\nabla_Y X, K\rangle = -\langle Y, \bar\nabla_X K\rangle + \langle X, \bar\nabla_Y K\rangle. \qquad (17)$$

Suppose first that $\langle X, W\rangle = \langle Y, W\rangle = 0$. The above expression becomes

$$-\langle Y, \phi X\rangle + \langle X, \phi Y\rangle = 0,$$

which shows that $[X,Y] \in K^\perp$. The same conclusion is valid when $X, Y$ are multiples of $W$.

On the other hand, if $X \in W^\perp$ and $Y = cW$, the corresponding expression for $\langle [X,Y], K\rangle$ is

$$-\langle Y, \phi X\rangle + \langle X, \psi Y\rangle = c(\psi - \phi)\langle X, W\rangle,$$

which vanishes again, because $X$ and $W$ are orthogonal. Thus, we also have in this case that $[X,Y] \in K^\perp$, and $K^\perp$ is involutive.

Now, let $M$ be a leaf of the foliation determined by $K^\perp$. By (16), we have

$$\bar\nabla_X N = \frac{\phi}{|K|} X \quad \text{for } \langle X, W\rangle = \langle X, K\rangle = 0,$$

so that the $(n-1)$-dimensional distribution $K^\perp \cap W^\perp$ satisfies (3) and $M$ is $(n-1)$-umbilical, as desired.

The following lemma establishes that in fact, the $n-1$ equal principal curvatures of a leaf are constant.

**Lemma 3.3.** *The functions $|K|^2$, $\phi$ and each of the $(n-1)$ equal principal curvatures are constant along each connected leaf of the foliation determined by $K^\perp$.*

**Proof:** We fix an orthonormal frame $E_1, \ldots, E_{n-1}, W, N$ such that the vector fields $E_1, \ldots, E_{n-1}$ span $K^\perp \cap W^\perp$.

First we calculate the coefficients of the gradient of $|K|^2 = -\langle K, K\rangle$ with respect to this frame:

$$\begin{aligned}\langle \text{grad } |K|^2, E_i\rangle &= -2\langle \bar{\nabla}_{E_i} K, K\rangle = -2\langle \phi E_i, K\rangle = 0, \quad i = 1, \dots, n-1;\\ \langle \text{grad } |K|^2, W\rangle &= -2\langle \psi W, K\rangle = 0;\\ \langle \text{grad } |K|^2, N\rangle &= -2\langle \phi N, K\rangle = 2\phi|K|.\end{aligned}$$

From these equations we obtain that $|K|^2$ is constant along each connected leaf and

$$\text{grad}|K|^2 = 2\phi K.$$

The Hessian of $|K|^2$ is given by (see [13], p. 86, for example):

$$\begin{aligned}\text{Hess}|K|^2(U, V) &= \langle \bar{\nabla}_U(\text{grad}|K|^2), V\rangle\\ &= \langle \bar{\nabla}_U(2\phi K), V\rangle = 2\langle (U\phi)K + \phi\bar{\nabla}_U K, V\rangle,\end{aligned}$$

so that

$$\text{Hess}|K|^2(U, V) = 2\{(U\phi)\langle K, V\rangle + \phi\langle \bar{\nabla}_U K, V\rangle\}. \tag{18}$$

We use this formula and the partial conformality of $K$ to calculate the matrix of the Hessian with respect to the frame. The coefficients we are interested in are the following:

$$\begin{aligned}\tfrac{1}{2}\text{Hess}|K|^2(E_i, N) &= (E_i\phi)\langle K, N\rangle + \phi\langle \bar{\nabla}_{E_i} K, N\rangle\\ &= (E_i\phi)\langle K, N\rangle + \phi^2\langle E_i, N\rangle\\ &= (E_i\phi)\langle K, N\rangle;\\ \tfrac{1}{2}\text{Hess}|K|^2(N, E_i) &= (N\phi)\langle K, E_i\rangle + \phi\langle \bar{\nabla}_N K, E_i\rangle\\ &= (N\phi)\langle K, E_i\rangle + \phi^2\langle K, E_i\rangle = 0,\end{aligned}$$

for $i = 1, \dots, n-1$. As the Hessian is symmetric, we must have $E_i\phi = 0$. Similarly, $W\phi = 0$, which shows that $\phi$ is constant along each connected leaf of the foliation determined by $K^\perp$.

The conditions (16) over $E_1, \dots, E_{n-1}, W$ imply that they correspond to the principal directions on $M$; each principal curvature is given by

$$\begin{aligned}\kappa_i &= -\langle \bar{\nabla}_{E_i} N, E_i\rangle = -\tfrac{\phi}{|K|}, \quad i = 1, \dots, n-1,\\ \kappa_n &= -\langle \bar{\nabla}_W N, E_0\rangle = -\tfrac{\psi}{|K|}.\end{aligned} \tag{19}$$

As we have shown, $\phi$ and $|K|$ are constant along $M$. Hence the principal curvatures $\kappa_1, \dots, \kappa_{n-1}$ are constant as well.

# 4 Spacelike $(n-1)$-umbilical foliations in space forms

To finish this paper we return to the study of $(n-1)$-umbilical spacelike hypersurfaces, now from the point of view of foliations. We will give examples in the Lorentzian space forms $\mathbb{Q}_1^{n+1}(c)$, $c \geq 0$.

In the case of the Lorentz-Minkowski space $\mathbb{R}_1^{n+1}$, let $K$ be the vector field

$$K(x_1, x_2, \ldots, x_{n+1}) = \frac{1}{\sqrt{x_1^2 - x_2^2}}(x_1, x_2, 0, \ldots, 0)$$

defined in the open set of $\mathbb{R}_1^{n+1}$ given by $x_1^2 - x_2^2 > 0$. Note that $K$ is a timelike unit vector field. If $e_1, \ldots, e_{n+1}$ denote the canonical basis of the tangent space to $\mathbb{R}_1^{n+1}$ at a point of this open set, then

$$\bar{\nabla}_{e_3} K = \cdots = \bar{\nabla}_{e_{n+1}} K = 0,$$

but also $\bar{\nabla}_K K = 0$. It is straightforward to prove that

$$\bar{\nabla}_{x_2 e_1 + x_1 e_2} K = \frac{1}{\sqrt{x_1^2 - x_2^2}}(x_2 e_1 + x_1 e_2).$$

These calculations show that $K$ is a closed partially conformal timelike vector field, with $\phi = 0$ and $\psi = 1/\sqrt{x_1^2 - x_2^2}$. Its associated foliation in the open set $x_1^2 - x_2^2 > 0$ is given by the hypersurfaces $\mathbb{H}^1(c) \times \mathbb{R}^{n-1}$, $c < 0$, defined as

$$\mathbb{H}^1(c) \times \mathbb{R}^{n-1} = \left\{ (x, y) \in \mathbb{R}_1^{n+1} = \mathbb{R}_1^2 \times \mathbb{R}^{n-1} \,\middle|\, x_1^2 - x_2^2 = -\frac{1}{c} \right\},$$

which were already mentioned in [8]. Analogously, we may decompose $\mathbb{R}_1^{n+1}$ as $\mathbb{R}_1^n \times \mathbb{R}$ and define the vector field

$$K(x, y) = \frac{1}{\sqrt{\langle x, x \rangle}}(x, 0),$$

which is a closed partially conformal timelike vector field defined in the open set $\langle x, x \rangle > 0$; the associated foliation of this open set is given by the hypersurfaces

$$\mathbb{H}^{n-1}(c) \times \mathbb{R} = \left\{ (x, y) \in \mathbb{R}_1^n \times \mathbb{R} \,\middle|\, \langle x, x \rangle = -\frac{1}{c} \right\}.$$

In the case of the de Sitter space, we recall the definition of a hyperbolic cylinder given in (5), for $k = n-1$ and $\rho > 0$:

$$M = \{ (x_1, \ldots, x_{n+2}) \in \mathbb{S}_1^{n+1} \mid -x_1^2 + x_2^2 + \cdots + x_n^2 = -\sinh^2 \rho \}.$$

Varying $\rho$, we get a family of hyperbolic cylinders foliating the open set

$$\{ (x_1, \ldots, x_{n+2}) \in \mathbb{S}_1^{n+1} \mid -x_1^2 + x_2^2 + \cdots + x_n^2 < 0 \}.$$

It was observed by Montiel in [9] that the vector field $N$ given by

$$N(p) = \frac{1}{\sinh \rho \cosh \rho}(x_1, \ldots, x_n, 0, 0) + (\tanh \rho) p$$

is a unit normal vector field for $M$ and he used it to prove that the principal curvatures are $\coth \rho$ and $\tanh \rho$ with multiplicities $(n-1)$ and 1, respectively. In fact, if we take the canonical basis $e_1, \ldots, e_{n+2}$ in the ambient space $\mathbb{R}_1^{n+2}$, it is easy to see that the tangent space $T_pM$ of $M$ at $p$ is spanned by the vectors

$$e_1 + e_2, e_1 + e_3, \ldots, e_1 + e_n \text{ and } e_{n+1} - e_{n+2}$$

and we have

$$\nabla_{e_1+e_i} N = (\coth \rho)(e_1 + e_i), \text{ for } i = 2, \ldots, n;$$

and

$$\nabla_{e_{n+1}-e_{n+2}} N = (\tanh \rho)(e_{n+1} - e_{n+2}),$$

which shows that $N$ is a closed partially conformal timelike vector field defined in the open subset of the de Sitter space here considered.

# References


[1] Akutagawa, K., *On spacelike hypersurfaces with constant mean curvature in the de Sitter space*, Math. Z. 196, (1987), 13–19.

[2] Brasil Jr., A.; Colares, A. G.; Palmas, O., *Complete spacelike hypersurfaces with constan mean curvature in the de Sitter space: A gap theorem*, Illinois J. Math. 47 (3), (2003), 847–866.

[3] Dajczer, M.; Florit, L., *On Chen's basic equality*, Illinois J. Math. 42 (1), (1998), 97–106.

[4] Dajczer, M.; Nomizu, K., *On flat surfaces in* $\mathbb{S}^3_1$ *and* $\mathbb{H}^3_1$, in Manifolds and Lie Groups, in honor of Y. Matsushima, Birkhäuser, Boston, (1981), 71–88.

[5] do Carmo, M. P.; Dajczer, M., *Rotational hypersurfaces in spaces of constant curvature*, Trans. Amer. Math. Soc. 277 (2), (1983), 685–709.

[6] Gliklikh, Y. E.; Morozova, L. A., *Conditions for global existence of solutions of ordinary differential, stochastic differential, and parabolic equations*, Int. J. Math. Math. Sci. 2004 (17), (2004), 901–912.

[7] Goddard, A. J., *Some remarks on the existence of spacelike hypersurfaces of constant mean curvature*, Math. Proc. Cambridge Phil. Soc. 82, (1977), 489–495.

[8] Ki, U-H.; Kim, H-J.; Nakagawa, H., *On spacelike hypersurfaces with constant mean curvature of a Lorentz space form*, Tokyo J. Math. 14 (1), (1991), 205–216.

[9] Montiel, S., *An integral inequality for complete spacelike hypersurfaces in De Sitter space and applications to the case of constant mean curvature*, Indiana Univ. Math. J., 37 (4), (1988), 909–917.

[10] Montiel, S., *A characterization of hyperbolic cylinders in the De Sitter space*, Tôhoku Math. J., 48, (1996), 23–31.

[11] Montiel, S., *Uniqueness of spacelike hypersurfaces of constant mean curvature in foliated spacetimes*, Math. Ann. 314, (1999), 529–553.

[12] Mori, H., *Rotational surfaces in a pseudo-Riemannian* 3*-sphere*. Tôhoku Math. J., 38, (1986), 29–36.

[13] O'Neill, B., *Semi-Riemannian Geometry*, Academic Press, (1983).

[14] Ramanathan, J., *Complete spacelike hypersurfaces of constant mean curvature in de Sitter space*, Indiana Univ. Math. J. 36 (2), (1987), 349–359.

[15] Tanno, S., *Partially conformal transformations with respect to* $(m-1)$*-dimensional distributions of* $m$*-dimensional Riemannian manifolds*, Tôhoku Math. J., 17, (1965), 358–409.

[16] Tanno, S., *Partially conformal transformations with respect to* $(m-1)$*-dimensional distributions of* $m$*-dimensional Riemannian manifolds, II*, Tôhoku Math. J., 18, (1966), 378–392.

Departamento de Matemática
Universidade Federal do Ceará
CEP 60.455-760 Fortaleza CE, Brazil
*E-mail*: gcolares@mat.ufc.br

Departamento de Matemáticas
Facultad de Ciencias, UNAM
México 04510 DF México
*E-mail*:
oscar.palmas@ciencias.unam.mx