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A NOTE ON THE DIRICHLET PROBLEM FOR THE MINIMAL SURFACE EQUATION IN NONCONVEX PLANAR DOMAINS

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Dedicated to Professor Manfredo do Carmo on the occasion of his 80^th birthday

Abstract

Given a bounded smooth domain Ω of $\mathbb{R}^2$ satisfying the exterior circle condition with radius r and a smooth boundary data φ on $\partial\Omega$, we prove that if r is bigger than a constant (explicitly calculated) depending only on the C^2 norm of φ then the Dirichlet problem for the minimal surface equation for Ω and φ has a solution. Since the condition on r is trivially satisfied if the domain is convex, our result generalizes the classical theorem of R. Finn [F].

Let Ω be a $C^{2,\alpha}$ bounded open domain in $\mathbb{R}^2$ satisfying the exterior circle condition of radius r, that is, given $p \in \partial\Omega$ there is a circle of radius r tangent to $\partial\Omega$ at p and contained in $\mathbb{R}^2 \backslash \Omega$.

Given $\varphi \in C^{2,\alpha}\left(\overline{\Omega}\right)$, set

$$\begin{aligned} M &= \max_{x\in\Omega} \varphi(x) - \min_{x\in\Omega} \varphi(x) \\ B &= \sup_{x\in\Omega} |D\varphi(x)| \\ A &= \sup_{x\in\Omega} \left|D^2\varphi(x)\right| \end{aligned}$$

where $\left|D^2\varphi\right| = |D_{11}\varphi| + |D_{22}\varphi| + |D_{12}\varphi|$.

We prove:

Theorem *If*

$$r \geq \max\left\{e^{4M\left(52A+50AB^2+10B^2+5\right)} - 1, 1\right\} \tag{1}$$

then the Dirichlet problem for the minimal surface equation

$$\begin{cases} M[u] := \left(1+|Du|^2\right)\Delta u - \sum\limits_{i,j=1}^{2} D_i u D_j u D_{ij} u = 0 \\ u|_{\partial\Omega} = \varphi. \end{cases}, \tag{2}$$

has an unique solution.

Since the minimality of a graph is invariant by homotheties, we may apply the above result to treat the case $0 < r < 1$ after rescaling the problem by the factor $1/r$.

If Ω is convex then we may take $r = \infty$ so that the above theorem recovers the classical result of R. Finn (see [F]) for smooth boundary data.

Proof. We may assume that $\min \varphi = 0$. Choose $p \in \partial\Omega$. Let $p_0 = (p_1, p_2)$ be the center of the circle tangent to $\partial\Omega$ at p, with radius r and contained in $\mathbb{R}^2 \backslash \Omega$, and set

$$d(x) = |x - p_0| - r.$$

Define $w \in C^{2,\alpha}\left(\overline{\Omega}\right)$ by

$$w(x) = \varphi(x) + \psi(d(x)),$$

where

$$\psi(d) = \delta \ln(bd + 1) \tag{3}$$

and δ, b are positive constants to be determined. We will only prove that w is an upper barrier at p for some choices of δ and b. For a lower barrier, note that if w is an upper barrier for $-\varphi$ at p, then $-w$ is a lower barrier for φ at p.

We have

$$\begin{aligned} D_i d(x) &= \frac{x_i - p_i}{|x - p_0|}, \\ D_{ij} d &= \frac{1}{|x - p_0|}\left(\delta_{ij} - \frac{(x_i - p_i)(x_j - p_j)}{|x - p_0|^2}\right), \\ \Delta d(x) &= \operatorname{div}\left(D_1 d(x), D_2 d(x)\right) = \frac{1}{|x - p_0|} \end{aligned}$$

so that

$$D_i\psi(d(x)) = \psi'(d(x))\frac{x_i - p_i}{|x-p_0|},$$

$$\begin{aligned} D_{ij}\psi(d(x)) &= \psi''(d(x))D_jd(x)D_id(x) + \psi'(d(x))D_{ij}d(x) \\ &= \psi''(d(x))\frac{(x_i-p_i)(x_j-p_j)}{|x-p_0|^2} \\ &\quad + \frac{\psi'(d(x))}{|x-p_0|}\left(\delta_{ij} - \frac{(x_i-p_i)(x_j-p_j)}{|x-p_0|^2}\right) \end{aligned}$$

and

$$\Delta\psi(d(x)) = \psi''(d(x)) + \frac{\psi'(d(x))}{|x-p_0|}.$$

Using the above equalities we obtain

$$\begin{aligned} M[w] &= \Delta\varphi + (D_1w)^2 D_{22}\varphi + (D_2w)^2 D_{11}\varphi - 2D_1wD_2wD_{12}\varphi \\ &\quad + \psi''(d)) + \frac{\psi'(d)}{|x-p_0|} + |Dw|^2 \psi''(d) \\ &\quad - \left(\psi''(d) - \frac{\psi'(d)}{|x-p_0|}\right)\left\langle (D_1w, D_2w), \left(\frac{x_1-p_1}{|x-p_0|}, \frac{x_2-p_2}{|x-p_0|}\right)\right\rangle^2 \end{aligned}$$

and the estimate

$$M[w] \le 2A\left(1 + B^2 + \psi'(d)^2\right) + \frac{\psi'(d)}{r}\left(1 + 2B^2\right) + \frac{2\psi'(d)^3}{r} + \psi''(d).$$

From (3) we obtain

$$\begin{aligned} M[w] &\le 2A\left(1 + B^2 + \delta^2\frac{b^2}{(bd+1)^2}\right) + \frac{\delta b}{r(bd+1)}\left(1+2B^2\right) \\ &\quad + \frac{2\delta^3b^3}{r(bd+1)^3} - \frac{\delta b^2}{(bd+1)^2} \le 2A\left(1 + B^2 + \frac{\delta^2 b^2}{(bd+1)^2}\right) \\ &\quad + \frac{\delta b}{r(bd+1)}\left(1+2B^2\right) - \frac{\delta b^2}{(bd+1)^2}\left(1 - \frac{2\delta^2 b}{r(bd+1)}\right). \end{aligned}$$

The last term above is negative if and only if

$$\frac{b}{bd+1} \le \frac{r}{2\delta^2}.$$

This inequality is satisfied if one chooses $b = r/\left(4\delta^2\right)$. With this choice of b we obtain

$$
\begin{aligned}
M[w] &\leq 2A\left(1+B^2+\frac{1}{16\delta^2}\frac{r^2}{\left(\frac{1}{4\delta^2}dr+1\right)^2}\right)+\frac{1}{4\delta}\frac{1}{\frac{1}{4\delta^2}dr+1}\left(1+2B^2\right) \\
&\quad -\frac{1}{32\delta^3}\frac{r^2}{\left(\frac{1}{4\delta^2}dr+1\right)^2} \\
&= \frac{1}{2\left(4\delta^2+dr\right)^2}\left(4AB^2d^2r^2+32AB^2dr\delta^2+4B^2dr\delta\right. \\
&\quad +64AB^2\delta^4+16B^2\delta^3+4Ad^2r^2+32Adr\delta^2+2dr\delta \\
&\quad \left.+4Ar^2\delta^2-r^2\delta+64A\delta^4+8\delta^3\right)
\end{aligned}
$$

We then have $M[w] \leq 0$ if

$$
\begin{aligned}
&4AB^2d^2r^2+32AB^2dr\delta^2+4B^2dr\delta+64AB^2\delta^4+16B^2\delta^3+4Ad^2r^2 \\
&+32Adr\delta^2+2dr\delta+4Ar^2\delta^2-r^2\delta+64A\delta^4+8\delta^3 \\
= \;&\left(4AB^2r^2+4Ar^2\right)d^2+\left(32ArB^2\delta^2+4rB^2\delta+32Ar\delta^2+2r\delta\right)d \\
&+64AB^2\delta^4+16B^2\delta^3+4Ar^2\delta^2-r^2\delta+64A\delta^4+8\delta^3 \leq 0
\end{aligned}
$$

For $0 < d \leq \delta$ we have

$$
\begin{aligned}
&\left(4AB^2r^2+4Ar^2\right)d^2+\left(32ArB^2\delta^2+4rB^2\delta+32Ar\delta^2+2r\delta\right)d \\
&+64AB^2\delta^4+16B^2\delta^3+4Ar^2\delta^2-r^2\delta+64A\delta^4+8\delta^3 \\
\leq \;&\left(4AB^2r^2+4Ar^2\right)\delta^2+\left(32ArB^2\delta^2+4rB^2\delta+32Ar\delta^2+2r\delta\right)\delta \\
&+64AB^2\delta^4+16B^2\delta^3+4Ar^2\delta^2-r^2\delta+64A\delta^4+8\delta^3 \\
= \;&\left(64AB^2+64A\right)\delta^4+\left(32Ar+16B^2+32AB^2r+8\right)\delta^3 \\
&+\left(4AB^2r^2+4B^2r+8Ar^2+2r\right)\delta^2+\left(-r^2\right)\delta \\
= \;&\delta\left[\left(64AB^2+64A\right)\delta^3+\left(32Ar+16B^2+32AB^2r+8\right)\delta^2\right. \\
&\left.+\left(4AB^2r^2+4B^2r+8Ar^2+2r\right)\delta+\left(-r^2\right)\right]
\end{aligned}
$$

Choosing $\delta \leq 1$, we obtain that $M[w] \leq 0$ for $0 < d \leq \delta$ if

$$\begin{aligned} & \left(64AB^2+64A\right)\delta^3+\left(32Ar+16B^2+32AB^2r+8\right)\delta^2 \\ & +\left(4AB^2r^2+4B^2r+8Ar^2+2r\right)\delta+\left(-r^2\right) \\ \leq \; & \left(64AB^2+64A\right)\delta+\left(32Ar+16B^2+32AB^2r+8\right)\delta \\ & +\left(4AB^2r^2+4B^2r+8Ar^2+2r\right)\delta-r^2\leq 0. \end{aligned}$$

It follows that $M\left[w\right]\leq 0$ for $0<d\leq\delta$ if

$$\delta\leq\frac{r^2}{\left(4AB^2+8A\right)r^2+\left(32A+32AB^2+4B^2+2\right)r+64A+64AB^2+16B^2+8}.$$

Noting that the function

$$f(r)=\frac{r^2}{\left(4AB^2+8A\right)r^2+\left(32A+32AB^2+4B^2+2\right)r+64A+64AB^2+16B^2+8}$$

is increasing on r and $r\geq 1$, we have

$$\frac{1}{104A+100AB^2+20B^2+10}=f(1)\leq f(r)$$

so that $M\left[w\right]\leq 0$ if $0<d\leq\delta$ for

$$\delta=\frac{1}{104A+100AB^2+20B^2+10}. \tag{4}$$

In sum: Defining δ by (4), taking $b=r/\left(4\delta^2\right)$, we have $M[w]\leq 0$ on $\mathcal{N}_p$ where

$$\mathcal{N}_p=\left\{x\in\Omega\mid 0\leq d(x)\leq\delta\right\}.$$

Thus, to guarantee that w is a local barrier from above for M on $\mathcal{N}_p$ the function w must satisfy the a priori height estimate

$$w|_{\partial\mathcal{N}_p}\geq u|_{\partial\mathcal{N}_p} \tag{5}$$

where u is a solution of $M[u]=0$ and $u|_{\partial\Omega}=\varphi$.

Note that with the choices above

$$\psi(d) = \frac{1}{104A + 100AB^2 + 20B^2 + 10} \ln \left(\frac{r\left(104A + 100AB^2 + 20B^2 + 10\right)^2 d}{4} + 1 \right)$$

At $\partial\mathcal{N}_p \cap \partial\Omega$ we have $u = \varphi$ so that (5) is satisfied at these points. By the maximum principle $\sup |u| \leq M$ so that, at $\partial\mathcal{N}_p \backslash \partial\Omega$ we have

$$w(x) = \psi(\delta) + \varphi(x) \geq \psi(\delta) - M.$$

Then (5) is satisfied at $\partial\mathcal{N}_p \backslash \partial\Omega$ if $\psi(\delta) \geq 2M$, which is the case if r satisfies (1).

Observing now that if condition (1) is satisfied for a given φ it is also satisfied for $t\varphi$ for any $t \in [0,1]$, we may conclude the proof of the theorem using the continuity method: Setting

$$V = \{t \in [0,1] \mid \exists u_t \in C^{2,\alpha}\left(\overline{\Omega}\right) \text{ such that } M\left[u_t\right] = 0,\ u_t|_{\partial\Omega} = t\varphi\}$$

we have $V \neq \varnothing$ since $t = 0 \in V$; moreover, V is open by the implicit function theorem. From the barriers above we obtain a priori uniform C^1 estimates for the family of Dirichlet problems $M\left[u_t\right] = 0$, $u_t|_{\partial\Omega} = t\varphi$, guaranteeing that V is closed ([GT]), that is, $V = [0,1]$.

The uniqueness of the solution is a consequence of the maximum principle for the difference of two solutions of (2).

This concludes with the proof of the theorem.

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