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ON THE ISOTROPIC REDUCTION METHOD AND THE MASLOV INDEX

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Abstract

We study the Maslov index of continuous paths in the Grassmannian Lagrangian using an isotropic reduction of the symplectic space, and we discuss a few applications.

1 Introduction

The Maslov index is a semi-integer invariant associated to continuous paths in the Lagrangian Grassmannian of a symplectic space (see [11, 12]), and it is one of the main tools employed in Morse theory for (periodic) solutions of Hamiltonian systems. In semi-Riemannian geometry, the Maslov index is usually associated to geodesics, and its value gives an algebraic counting of the conjugate or focal points along the geodesic. Let us observe here that the notion of Maslov index requires the choice of a fixed reference Lagrangian L_0. Usually, in the case of a Hamiltonian system in the cotangent bundle TM^* of a smooth manifold M, L_0 represents the vertical subspace of $T(TM^*)$. In the case of focal points along semi-Riemannian geodesics, the Lagrangian L_0 encodes the information on the tangent space of the initial orthogonal submanifold and on its second fundamental form in the direction of the geodesic.

It has been recently observed (see [1]), that in the case of horizontal geodesics in the total space of a semi-Riemannian submersion, the corresponding path in the Grassmannian Lagrangian (up to a continuous change of symplectic coordinates) consists of Lagrangian spaces that contain a fixed isotropic subspace. In this situation, the isotropic space represents the vertical Jacobi fields along

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the horizontal geodesic, that do not give any contribution to the Maslov index. It can be easily proved that the Maslov index of a path ℓ of Lagrangians that contain a fixed isotropic space S, relatively to a reference Lagrangian L_0 which is *decomposable* with respect to S, equals the Maslov index of the path $\widetilde{\ell}$ in the isotropic reduction space $S^{\perp}/S$ obtained by taking quotients $\ell(t)/S$. If L_0 is the Lagrangian associated to the vertical distribution and its second fundamental form, which is one of the two O'Neill tensors of the submersion (see [7, 8]), then L_0 is decomposable, and in this way one can obtain the equality between the focal index of the horizontal geodesic and the conjugate index of its projection onto the underlying base manifold of the submersion.

There are interesting situations where the decomposability property of the reference Lagrangian L_0 is not satisfied, and thus the equality between the Maslov index of the path and the Maslov index of its isotropic reduction requires more work. In this paper we develop a different technique that allows to get isotropic reduction under no assumption on the reference Lagrangian L_0 (Theorem 2.11). The proof of this result requires more involved direct calculations of the Maslov index in suitable local charts (Lemma 2.9), and the continuity of the isotropic reduction procedure is established by equivariance with respect to a smooth transitive action (Lemma 2.7 and Lemma 2.10).

We discuss a few applications of the result. A first immediate (and simple) corollary is obtained by applying the symplectic reduction in the calculation of the Hörmander index and of the Kashiwara index (Corollary 2.13 and Corollary 2.14). More interesting, the isotropic reduction result is employed to obtain an estimate on the difference of Maslov indices of a Lagrangian path relatively to the choice of two distinct reference Lagrangians (Proposition 3.7). This provides an extension of [5, Theorem 1.1, Proposition 4.1], and it aims at comparison results for conjugate or focal points along a semi-Riemannian geodesic. It should be observed that several different conventions can be made concerning the definition of Maslov index for paths whose endpoints are not transversal to the reference Lagrangian L_0. Here we are using the same convention as in [11], where the contribution of the endpoints is computed as one half of the signature of a certain bilinear form. In [5] the *focal index* of a Jacobi equation, which is an *integer valued invariant*, is computed by considering the total contribution of the endpoints. This explains the presence of the extra term $\epsilon \in \{0, \frac{1}{2}\}$ in our estimate (3.7).

The interested reader will find more applications to the study of conjugate and focal points in semi-Riemannian geometry in [4].

2 Isotropic reduction and Maslov index

The aim of this section is to prove an equality relating the Maslov index of a continuous path ℓ of Lagrangian subspaces of a symplectic space with the Maslov index of the path obtained as a quotient of ℓ by a fixed isotropic subspace S contained in $\ell(t)$ for all t. The Maslov index is an invariant associated to the path ℓ and to the choice of a fixed Lagrangian L_0. We will consider the case when L_0 and S are in general position, with special interest in two situations: when L_0 contains S, and when L_0 has trivial intersection with S.

2.1 Preliminaries

Let us consider a symplectic space (V, ω), with $\dim(V) = 2n$; we will denote by $\mathrm{Sp}(V, \omega)$ the *symplectic group* of (V, ω), which is the closed Lie subgroup of $\mathrm{GL}(V)$ consisting of all isomorphisms that preserve ω. A subspace $X \subset V$ is *isotropic* if the restriction of ω to $X \times X$ vanishes identically; an n-dimensional (i.e., maximal) isotropic subspace L of V is called a *Lagrangian subspace*. We denote by Λ the Lagrangian Grassmannian of (V, ω), which is the collection of all Lagrangian subspaces of (V, ω), and is a compact differentiable manifold of dimension $\frac{1}{2}n(n+1)$. A real-analytic atlas of charts on Λ is given as follows. Given a Lagrangian decomposition (L_0, L_1) of V, i.e., $L_0, L_1 \in \Lambda$ are transverse Lagrangians, so that $V = L_0 \oplus L_1$, then denote by $\Lambda^0(L_1)$ the open and dense subset of Λ consisting of all Lagrangians L transverse to L_1. A diffeomorphism

φ_{L_0,L_1} from $\Lambda^0(L_1)$ to the vector space $\mathrm{B}_{\mathrm{sym}}(L_0)$ of all symmetric bilinear forms on L_0 is defined by $\varphi_{L_0,L_1}(L) = \omega(T\cdot,\cdot)|_{L_0\times L_0}$, where $T : L_0 \to L_1$ is the unique linear map whose graph in $L_0 \oplus L_1 = V$ is L. The kernel of $\varphi_{L_0,L_1}(L)$ is the space $L \cap L_0$; moreover, the differential $\mathrm{d}\varphi_{L_0,L_1}(L_0) : T_{L_0}\Lambda \to \mathrm{B}_{\mathrm{sym}}(L_0)$ does not depend on the choice of $L_1 \in \Lambda^0(L_0)$, so that $T_{L_0}\Lambda$ has a canonical identification with $\mathrm{B}_{\mathrm{sym}}(L_0)$.

Given any two Lagrangian decompositions (L_0, L_1) and (L_0', L_1') of V, any isomorphism from L_0 to L_0' extends to a symplectomorphism T of V with the property that $T(L_1) = L_1'$. Let us recall a few notions related to symmetric bilinear forms. Given a symmetric bilinear form B on a (finite dimensional) real vector space W, the *index* of B is defined to be the dimension of a maximal subspace of W on which B is negative definite. The coindex of B is the index of $-B$, and the *signature* of B, denoted by $\mathrm{sign}(B)$ is defined to be the difference coindex minus index. If $W = W_1 + W_2$, the sum being not necessarily direct, and W_1, W_2 are B-orthogonal subspaces of W, i.e., $B(w_1, w_2) = 0$ for all $w_i \in W_i$, then $\mathrm{sign}(B)$ equals the sum of the signatures of the restrictions of B to $W_1 \times W_1$ and to $W_2 \times W_2$. If $f : W' \to W$ is a surjective linear map, then the signature of the *pull-back* $f^*(B)$ of B by f, which is defined by $f^*(B)(v_1, v_2) = B\big(f(v_1), f(v_2)\big)$ for all $v_1, v_2 \in W'$, equals the signature of B.

Let us now briefly recall the notion of Maslov index for a continuous path $\ell : [a, b] \to \Lambda$. For a fixed Lagrangian $L_0 \in \Lambda$, the L_0-*Maslov index* $\mu_{L_0}(\ell)$ of ℓ is the half-integer characterized by the following properties:

(a) μ_{L_0} is fixed-endpoint homotopy invariant;

(b) μ_{L_0} is additive by concatenation;

(c) if $\ell\big([a, b]\big) \subset \Lambda^0(L_1)$ for some Lagrangian L_1 transverse to L_0, then

$$\mu_{L_0}(\ell) = \tfrac{1}{2}\mathrm{sign}\big[\varphi_{L_0,L_1}\big(\ell(b)\big)\big] - \tfrac{1}{2}\mathrm{sign}\big[\varphi_{L_0,L_1}\big(\ell(a)\big)\big]. \qquad (2.1)$$

If $\ell : [a, b] \to \Lambda$ is a curve of class C^1 and the symmetric bilinear form $\ell'(a) \in \mathrm{B}_{\mathrm{sym}}\big(\ell(a)\big)$ is nondegenerate on the (possibly trivial) intersection $\ell(a)\cap L_0$, then for $\varepsilon > 0$ small enough it is $\ell(t) \cap L_0 = \ell(a) \cap L_0$ for $t \in]0, \varepsilon]$, and

$$\mu_{L_0}\big(\ell|_{[a,a+\varepsilon]}\big) = \frac{1}{2}\mathrm{sign}\big(\ell'(a)|_{\ell(a)\cap L_0}\big).$$

Given any continuous path $\ell : [a, b] \to \Lambda$ and any two Lagrangians $L_0, L_0' \in \Lambda$, the difference $\mu_{L_0}(\ell) - \mu_{L_0'}(\ell)$ depends only on L_0, L_0' and the endpoints $\ell(a)$ and $\ell(b)$ of ℓ. This quantity will be denoted by $\mathfrak{q}\big(L_0, L_0'; \ell(a), \ell(b)\big)$, and it coincides (up to some factor which is irrelevant here) with the so called *Hörmander index*. The quantity:

$$\tau(L_0, L_1, L_2) = \mathfrak{q}(L_0, L_1; L_2, L_0) = -\mathfrak{q}(L_0, L_1; L_0, L_2)$$

coincides (again up to some factor) with the *Kashiwara index*. The Kashiwara index function determines completely the Hörmander index, by the identity:

$$\mathfrak{q}(L_0, L_1; L_0', L_1') = \tau(L_0, L_1, L_0') - \tau(L_0, L_1, L_1'), \quad \forall\, L_0, L_1, L_0', L_1' \in \Lambda, \tag{2.2}$$

which is proved easily using the concatenation additivity property of the Maslov index.

Given a subspace $S \subset V$, we will denote by $S^\perp$ the symplectic orthogonal space of S, which consists of all $v \in V$ such that $\omega(v, w) = 0$ for every $w \in S$. Then, a subspace S is isotropic if $S \subset S^\perp$, and it is Lagrangian if $S = S^\perp$. Given an isotropic subspace $S \subset V$, one has a natural symplectic form $\overline{\omega}$ on the quotient $S^\perp/S$, defined by setting $\overline{\omega}\big(v + S, w + S\big) = \omega(v, w)$, for all $v, w \in S^\perp$. The symplectic space $(S^\perp/S, \overline{\omega})$ will be called an *isotropic reduction* of (V, ω).

We will investigate the relations between Lagrangian decompositions and Maslov index in a symplectic space and in one of its isotropic reductions.

2.2 Lagrangian decompositions of an isotropic reduction

Isotropic subspaces can always be *enlarged* to Lagrangian subspaces, in the following strong way:

Lemma 2.1. *If $L_0 \subset V$ is a Lagrangian subspace, $S \subset V$ is an isotropic subspace and $L_0 \cap S = \{0\}$ then there exists a Lagrangian subspace $L \subset V$ containing S with $L_0 \cap L = \{0\}$.*

Proof: It suffices to show that if S is not Lagrangian then there exists an isotropic subspace $\widetilde{S}$ of V containing S with $L_0 \cap \widetilde{S} = \{0\}$ and $S \neq \widetilde{S}$. If we can find $v \in S^\perp$ with $v \notin L_0 + S$, then the isotropic subspace $\widetilde{S}$ can be obtained by setting $\widetilde{S} = S + \mathbb{R}v$. Thus, we have to show that $S^\perp$ is not contained in $L_0 + S$. But $S^\perp \subset L_0 + S$ implies $(L_0 + S)^\perp \subset (S^\perp)^\perp = S$, i.e., $L_0^\perp \cap S^\perp = L_0 \cap S^\perp \subset S$. Then $L_0 \cap S^\perp \subset L_0 \cap S = \{0\}$. Since S is not Lagrangian, $\dim(S^\perp) > n$, hence $L_0 \cap S^\perp \neq \{0\}$ and we obtain a contradiction.

□

Given any Lagrangian L, then $(L \cap S^\perp) + S$ is another Lagrangian. Namely, such space is clearly isotropic, moreover, its dimension is easily computed as follows:

$$\dim(L \cap S^\perp) = \dim\big((L + S)^\perp\big) = 2n - \dim(L + S) = n - \dim(S) + \dim(L \cap S),$$

and so:

$$\begin{aligned}\dim\big((L \cap S^\perp) + S\big) &= \dim(L \cap S^\perp) + \dim(S) - \dim(L \cap S \cap S^\perp)\\ &= \dim(L \cap S^\perp) + \dim(S) - \dim(L \cap S) = n.\end{aligned}$$

Let us recall the following result on the Kashiwara index from [6]:

Lemma 2.2. *Given any three Lagrangians $L_0, L_1, L_2 \in \Lambda$ and any isotropic subspace $S \subset (L_0 \cap L_1) + (L_0 \cap L_2) + (L_1 \cap L_2)$, then, denoting by $L_i^S = (L_i \cap S^\perp) + S$, $i = 0, 1, 2$, one has:*

$$\tau(L_0, L_1, L_2) = \tau(L_0^S, L_1^S, L_2^S).$$

Proof: See [6, Proposition 1.5.10].

□

There is a natural way of obtaining Lagrangian subspaces of an isotropic reduction using Lagrangian subspaces of V, described in the following:

Lemma 2.3. *Let S be an isotropic subspace of V and consider the quotient map $q : S^\perp \to S^\perp/S$ onto the symplectic space $(S^\perp/S, \overline{\omega})$.*

(a) If L is a Lagrangian subspace of V then $q(L \cap S^\perp)$ is a Lagrangian subspace of $S^\perp/S$. In particular, if L is a Lagrangian subspace of V containing S then L/S is a Lagrangian subspace of $S^\perp/S$.

(b) If (L_0, L_1) is a Lagrangian decomposition of V then the following two conditions are equivalent:

- *$L_1 \cap S = \{0\}$ and $\big(q(L_0 \cap S^\perp), q(L_1 \cap S^\perp)\big)$ is a Lagrangian decomposition of $S^\perp/S$;*
- *$\big((L_0 \cap S^\perp) + (L_1 \cap S^\perp)\big) \cap S = L_0 \cap S$.*

Proof: For part (a) it is immediate that $q(L \cap S^\perp)$ is isotropic. To compute the dimension of $q(L \cap S^\perp)$, observe that $q(L \cap S^\perp)$ is the image of the restriction of q to $L \cap S^\perp$ and that the kernel of such restriction is $L \cap S^\perp \cap S = L \cap S$. Thus:

$$\dim(L \cap S^\perp) = \dim(L \cap S) + \dim\big(q(L \cap S^\perp)\big). \tag{2.3}$$

But $L \cap S^\perp = (L + S)^\perp$ and therefore:

$$\dim(L \cap S^\perp) = 2n - \dim(L + S). \tag{2.4}$$

Combining (2.3) and (2.4) and using that $\dim(L+S)+\dim(L\cap S) = n+\dim(S)$ we get $\dim\big(q(L \cap S^\perp)\big) = n - \dim(S) = \frac{1}{2}\dim(S^\perp/S)$. This proves part (a). Part (b) follows from an immediate Linear Algebra argument.

□

If $T \in \mathrm{Sp}(V, \omega)$ is such that $T(S) \subset S$, and hence also $T(S^\perp) \subset S^\perp$, then the restriction $T|_{S^\perp} : S^\perp \to S^\perp$ induces an isomorphism $\overline{T} : S^\perp/S \to S^\perp/S$, i.e., we have a commutative diagram:

$$\begin{tikzcd} S^\perp \arrow[r, "T|_{S^\perp}"] \arrow[d, "q"'] & S^\perp \arrow[d, "q"] \\ S^\perp/S \arrow[r, "\overline{T}"'] & S^\perp/S \end{tikzcd} \tag{2.5}$$

where $q : S^\perp \to S^\perp/S$ denotes the quotient map. It is easy to see that $\overline{T}$ preserves $\overline{\omega}$, i.e., $\overline{T} \in \mathrm{Sp}(S^\perp/S, \overline{\omega})$. In fact, a more general statement holds:

Lemma 2.4. *Let $\ell \subset V$ be a Lagrangian subspace and let $S \subset \ell$ be any subspace. Consider the quotient symplectic form $\overline{\omega}$ on $S^\perp/S$; then, given any symplectomorphism $\overline{T}$ of $(S^\perp/S, \overline{\omega})$ with $\overline{T}(q(\ell)) = q(\ell)$, there exists a symplectomorphism T of (V, ω) such that $T(S) = S$ (hence also $T(S^\perp) = S^\perp$), $T(\ell) = \ell$, and such that* (2.5) *commutes.*

Proof: See for instance [9, Lemma 1.4.42, p. 39].

□

2.3 Isotropic reduction and Maslov index

In this section we consider a fixed $2n$-dimensional symplectic space (V, ω) and an isotropic subspace S of V; we will consider the symplectic form $\overline{\omega}$ on the quotient $S^\perp/S$.

Lemma 2.5. *If L_0 is a Lagrangian subspace of V then there exists a Lagrangian subspace L_1 of V with $L_0 \cap L_1 = \{0\}$ and:*

$$\big((L_0 \cap S^\perp) + (L_1 \cap S^\perp)\big) \cap S = L_0 \cap S. \tag{2.1}$$

Proof: Observe that the righthand side of (2.1) is a subspace of the lefthand side of (2.1), for any choice of L_1. Let S' be a subspace of S with:

$$S = (L_0 \cap S) \oplus S'.$$

Since S' is an isotropic subspace with $L_0 \cap S' = \{0\}$, by Lemma 2.1, there exists a Lagrangian subspace L of V containing S' with $L_0 \cap L = \{0\}$. Let $L_1 \in \Lambda^0(L_0)$ be such that the symmetric bilinear form $\varphi_{L,L_0}(L_1)$ in L is positive definite. To prove (2.1), let $v \in L_0 \cap S^\perp$, $w \in L_1 \cap S^\perp$ be fixed with $v + w \in S$. Write $v + w = u_1 + u_2$ with $u_1 \in L_0 \cap S$ and $u_2 \in S'$. The proof will be concluded if we show that $u_2 = 0$. Denote by T the linear map $T : L \to L_0$ whose graph in

$L \oplus L_0$ is L_1. We have:

$$L_1 \ni w = u_2 + (u_1 - v),$$

with $u_2 \in S' \subset L$ and $u_1 - v \in L_0$, so that $u_1 - v = T(u_2)$. Thus:

$$\varphi_{L,L_0}(L_1)(u_2, u_2) = \omega\big(T(u_2), u_2\big) = \omega(u_1 - v, u_2) = 0,$$

$u_1 \in S \subset S^\perp$, $v \in S^\perp$ and $u_2 \in S$. But $\varphi_{L,L_0}(L_1)$ is positive definite and therefore $u_2 = 0$.

□

Let us recall the following result concerning the smoothness of equivariant maps. Recall that if M and N are smooth manifolds endowed with a smooth (left) action of the Lie group G, then a map $\phi : M \to N$ is said to be G-*equivariant* if $\phi(g \cdot x) = g \cdot \phi(x)$ for all $x \in M$ and all $g \in G$.

Proposition 2.6. *Let M, N be manifolds and let G be a Lie group that acts differentiably on both M and N. If the action of G on M is transitive, then every equivariant map $\phi : M \to N$ is differentiable.*

Proof: See for instance [9, Corollary 2.1.10, p. 66].

□

Lemma 2.7. *The set:*

$$\big\{L \in \Lambda : L \cap S = \{0\}\big\} \tag{2.2}$$

is open in Λ and the map (recall part (b) of Lemma 2.3):

$$\big\{L \in \Lambda : L \cap S = \{0\}\big\} \ni L \longmapsto q(L \cap S^\perp) \in \Lambda(S^\perp/S) \tag{2.3}$$

is differentiable.

Proof: The set (2.2) is open in Λ because, by Lemma 2.1, it is equal to the union:

$$\bigcup_{\substack{\ell \in \Lambda \\ \ell \supset S}} \Lambda^0(\ell).$$

Let G be the closed (Lie) subgroup of $\mathrm{Sp}(V,\omega)$ consisting of those symplectomorphisms $T : V \to V$ such that $T(S) = S$. The canonical action of $\mathrm{Sp}(V,\omega)$ on Λ restricts to a differentiable action of G on (2.2). We also have a differentiable action of G on $\Lambda(S^\perp/S)$ given by:

$$G \times \Lambda(S^\perp/S) \ni (T, \widetilde{L}) \longmapsto \overline{T}(\widetilde{L}) \in \Lambda(S^\perp/S),$$

where $\overline{T}$ is the symplectomorphism induced by T on $S^\perp/S$ (see (2.5)). The map (2.3) is obviously equivariant. The conclusion will follow from Proposition 2.6 once we show that the action of G on (2.2) is transitive. To this aim, let L_1, L_2 be in (2.2). By Lemma 2.1 there exist Lagrangians L_1', L_2' containing S such that $L_1 \cap L_1' = \{0\}$ and $L_2 \cap L_2' = \{0\}$. Now choose an arbitrary isomorphism from L_1' to L_2' that preserves S and let T be a symplectomorphism of V that extends such isomorphism and such that $T(L_1) = L_2$.

□

Corollary 2.8. *Given Lagrangian subspaces L_0, ℓ of V with $S \subset \ell$ there exists a Lagrangian subspace L_1 of V with $L_0 \cap L_1 = \{0\}$, $\ell \cap L_1 = \{0\}$ and such that* (2.1) *holds.*

Proof: By Lemma 2.7 the set:

$$\big\{L \in \Lambda : L \cap S = \{0\} \text{ and } q(L \cap S^\perp) \cap q(L_0 \cap S^\perp) = \{0\}\big\} \tag{2.4}$$

is open, being the inverse image by the continuous map (2.3) of the open subset

$$\Lambda^0\big(q(L_0 \cap S^\perp)\big)$$

of the Lagrangian Grassmannian of $S^\perp/S$. By part (b) of Lemma 2.3 the Lagrangian L_1 whose existence is granted by Lemma 2.5 is in (2.4). Using the fact that the set of Lagrangians transverse to a given Lagrangian is open and dense, it follows that the intersection of (2.4) with $\Lambda^0(L_0) \cap \Lambda^0(\ell)$ is nonempty. The desired Lagrangian L_1 can be taken to be a member of such intersection.

□

Lemma 2.9. *Let (L_0, L_1) be a Lagrangian decomposition of V such that (2.1) holds, so that $L_1 \cap S = \{0\}$ and $(\widetilde{L}_0, \widetilde{L}_1) = \big(q(L_0 \cap S^\perp), q(L_1 \cap S^\perp)\big)$ is a Lagrangian decomposition of $S^\perp/S$ (recall part (b) of Lemma 2.3). Given a Lagrangian subspace ℓ of V containing S then:*

(a) $\ell \cap L_1 = \{0\}$ if and only if $q(\ell) \cap \widetilde{L}_1 = \{0\}$.

Assuming that a given Lagrangian ℓ containing S is transverse to L_1 then:

(b) The pull-back by the surjective map $q|_{L_0 \cap S^\perp} : L_0 \cap S^\perp \to \widetilde{L}_0$ of $\varphi_{\widetilde{L}_0,\widetilde{L}_1}\big(q(\ell)\big)$ is equal to the restriction of $\varphi_{L_0,L_1}(\ell)$ to $L_0 \cap S^\perp$.

(c) If $\pi : V \to L_0$ denotes the projection with respect to the decomposition $V = L_0 \oplus L_1$ then $L_0 = \pi(S) + (L_0 \cap S^\perp)$.

(d) If π is as in part (c) then the spaces $\pi(S)$ and $L_0 \cap S^\perp$ are orthogonal with respect to the symmetric bilinear form $\varphi_{L_0,L_1}(\ell)$.

(e) The restriction of the symmetric bilinear form $\varphi_{L_0,L_1}(\ell)$ to $\pi(S) \times \pi(S)$ is independent of ℓ.

Proof: Since ℓ contains S and $L_1 \cap S = \{0\}$ it follows that $q(\ell) \cap \widetilde{L}_1 = q(\ell) \cap q(L_1 \cap S^\perp) = \{0\}$ if and only if $\ell \cap (L_1 \cap S^\perp) = \{0\}$. Item (a) then follows by observing that ℓ is contained in $S^\perp$. Let $T : L_0 \to L_1$ be the linear map whose graph in $L_0 \oplus L_1$ is equal to ℓ and let $\widetilde{T} : \widetilde{L}_0 \to \widetilde{L}_1$ be the linear map whose graph in $\widetilde{L}_0 \oplus \widetilde{L}_1$ is equal to $q(\ell)$. Let $v \in L_0 \cap S^\perp$ be fixed. We have $v + T(v) \in \ell \subset S^\perp$ and thus $T(v) \in L_1 \cap S^\perp$. Therefore $q(v) \in \widetilde{L}_0$, $q\big(T(v)\big) \in \widetilde{L}_1$ and $q(v) + q\big(T(v)\big) \in q(\ell)$. This implies that $q\big(T(v)\big) = \widetilde{T}\big(q(v)\big)$. Now, given $w \in L_0 \cap S^\perp$ we have:

$$\begin{aligned}\varphi_{L_0,L_1}(\ell)(v,w) = \omega\big(T(v), w\big) = \overline{\omega}\big(q(T(v)), q(w)\big)\\ = \overline{\omega}\big(\widetilde{T}(q(v)), q(w)\big) = \varphi_{\widetilde{L}_0,\widetilde{L}_1}\big(q(\ell)\big)\big(q(v), q(w)\big),\end{aligned}$$

proving (b). To prove (c), we will show that $\dim\big(\pi(S) + (L_0 \cap S^\perp)\big) = n$. We have:

$$\dim\big(\pi(S) + (L_0 \cap S^\perp)\big) = \dim\big(\pi(S)\big) + \dim(L_0 \cap S^\perp) - \dim\big(\pi(S) \cap (L_0 \cap S^\perp)\big). \tag{2.5}$$

Since $S \cap L_1 = \{0\}$, we have:

$$\dim\big(\pi(S)\big) = \dim(S). \tag{2.6}$$

Moreover:

$$\begin{aligned} \dim(L_0 \cap S^\perp) = \dim\big((L_0 + S)^\perp\big) &= 2n - \dim(L_0 + S) \\ &= n - \dim(S) + \dim(L_0 \cap S). \end{aligned} \tag{2.7}$$

Let us now prove that:

$$\pi(S) \cap (L_0 \cap S^\perp) = L_0 \cap S. \tag{2.8}$$

Notice that combining (2.5), (2.6), (2.7) and (2.8) we will conclude the proof of part (c). Clearly $L_0 \cap S = \pi(L_0 \cap S) \subset \pi(S)$ and thus $L_0 \cap S \subset \pi(S) \cap (L_0 \cap S^\perp)$. Now let $v \in S$ be such that $\pi(v) \in L_0 \cap S^\perp$ and let us show that $\pi(v) \in S$. We have $v - \pi(v) \in L_1$, $v \in S \subset S^\perp$, $\pi(v) \in S^\perp$, so that $v - \pi(v) \in L_1 \cap S^\perp$; then:

$$v = \pi(v) + \big(v - \pi(v)\big) \in (L_0 \cap S^\perp) + (L_1 \cap S^\perp),$$

and it follows from (2.1) that $v \in L_0 \cap S$. Thus $\pi(v) = v \in S$. This proves (2.8) and concludes the proof of part (c). To prove part (d), pick $v \in S$, $w \in L_0 \cap S^\perp$ and let us show that $\varphi_{L_0,L_1}(\ell)\big(\pi(v), w\big) = 0$. Since $v \in \ell$ we can write $v = u + T(u)$, with $u \in L_0$; then $\pi(v) = u$. Now:

$$\varphi_{L_0,L_1}(\ell)\big(\pi(v), w\big) = \omega\big(T(u), w\big) = \omega\big(u + T(u), w\big) - \omega(u, w) = 0,$$

since $u + T(u) = v \in S$, $w \in S^{\perp}$ and $u, w \in L_0$. To prove (e), let $\ell, \ell' \in \Lambda$ be Lagrangians transverse to L_1 containing S. Set $T : L_0 \to L_1$ be the linear map whose graph is ℓ, and $T' : L_0 \to L_1$ be the linear map whose graph is ℓ'. The proof will be concluded if we show that T and T' agree on $\pi(S)$. Given $v \in S$, write $v = v_0 + v_1$, with $v_0 = \pi(v) \in \pi(S) \subset L_0$ and $v_1 \in L_1$. Since $v \in \ell$ and $v \in \ell'$ we have $T\big(\pi(v)\big) = v_1$ and $T'\big(\pi(v)\big) = v_1$. This concludes the proof.

□

We also need the following analogue of Lemma 2.7:

Lemma 2.10. *The set:*

$$\big\{L \in \Lambda : L \supset S\big\} \tag{2.9}$$

is a closed submanifold of Λ *and the map:*

$$\big\{L \in \Lambda : L \supset S\big\} \ni L \longmapsto L/S \in \Lambda(S^{\perp}/S) \tag{2.10}$$

is differentiable.

Proof: Let G be as in the proof of Lemma 2.7. Clearly the action of G on $\Lambda(V)$ preserves (2.9). We claim that the action of G on (2.9) is transitive. Namely, given L_1, L_2 in (2.9) then pick any isomorphism T from L_1 to L_2 that preserves S and choose any symplectomorphism of V that extends T. Clearly, (2.9) is equal to the intersection:

$$\bigcap_{v \in S} \big\{L \in \Lambda : L \ni v\big\}$$

and therefore it is closed. Thus, (2.9) is a closed orbit of the action of G on Λ, and therefore it is a smooth embedded submanifold of Λ (see [13, Theorem 2.9.7]). If we consider the action of G on $\Lambda(S^{\perp}/S)$ defined in the proof of Lemma 2.7 then clearly the map (2.10) is G-equivariant and therefore, by Proposition 2.6, it is differentiable.

□

We are finally ready for the main result of this section.

Theorem 2.11. *Let us assume that $\ell : [a,b] \to \Lambda$ is a continuous path; let S be an isotropic subspace of V such that $S \subset \ell(t)$ for all $t \in [a,b]$. Denote by $q : S^\perp \to S^\perp/S$ the quotient map. Let $\widetilde{\ell} : [a,b] \to \Lambda(S^\perp/S, \overline{\omega})$ be the continuous path given by $\widetilde{\ell}(t) = q\big(\ell(t)\big)$ and by $\widetilde{L}_0$ the Lagrangian*

$$\widetilde{L}_0 = q(L_0 \cap S^\perp) \in \Lambda(S^\perp/S, \overline{\omega}). \tag{2.11}$$

Then:

$$\mu_{L_0}(\ell) = \mu_{\widetilde{L}_0}(\widetilde{\ell}). \tag{2.12}$$

Proof: The continuity of the path $\widetilde{\ell}$ follows from Lemma 2.10. By the continuity of ℓ, Corollary 2.8 tells us that one can find a partition of the interval $[a,b]$, $a = a_0 < a_1 < \ldots < a_N = b$, and Lagrangians $L_1, \ldots, L_N \in \Lambda$ such that:

(i) $L_i \cap L_0 = \{0\}$ for all $i = 1, \ldots, N$;

(ii) $L_i \cap \ell(t) = \{0\}$ for all $t \in [a_{i-1}, a_i]$ and for all $i = 1, \ldots, N$;

(iii) $\big((L_0 \cap S^\perp) + (L_i \cap S^\perp)\big) \cap S = L_0 \cap S$ for all $i = 1, \ldots, N$.

Thus, each (L_0, L_i) is a Lagrangian decomposition of V for all i and, by part (b) of Lemma 2.3, setting $\widetilde{L}_i = q(L_i \cap S^\perp)$, $(\widetilde{L}_0, \widetilde{L}_i)$ is a Lagrangian decomposition of $S^\perp/S$. By (ii) above, $\ell|_{[a_{i-1},a_i]}$ has image in the domain of the chart φ_{L_0,L_i} for all i, and by part (a) of Lemma 2.9, $\widetilde{\ell}|_{[a_{i-1},a_i]}$ has image in the domain of the chart $\varphi_{\widetilde{L}_0,\widetilde{L}_i}$ for all i. In order to prove the theorem, it suffices to show that:

$$\begin{aligned}\tfrac{1}{2}\mathrm{sign}\big[\varphi_{L_0,L_i}\big(\ell(a_i)\big)\big] &- \tfrac{1}{2}\mathrm{sign}\big[\varphi_{L_0,L_i}\big(\ell(a_{i-1})\big)\big] \\ &= \tfrac{1}{2}\mathrm{sign}\big[\varphi_{\widetilde{L}_0,\widetilde{L}_i}\big(\widetilde{\ell}(a_i)\big)\big] - \tfrac{1}{2}\mathrm{sign}\big[\varphi_{\widetilde{L}_0,\widetilde{L}_i}\big(\widetilde{\ell}(a_{i-1})\big)\big],\end{aligned}$$

for all i. This follows easily from parts (b), (c), (d) and (e) of Lemma 2.9. Namely, for $t \in [a_{i-1}, a_i]$, the signature of $B_t = \varphi_{L_0,L_i}\big(\ell(t)\big)$ is given by the sum of the signatures of the restrictions of B_t to $\pi(S)$ and to $L_0 \cap S^\perp$, by part (c) and part (d) of Lemma 2.9. The signature of the restriction of B_t

to $L_0 \cap S^\perp$ equals the signature of $\varphi_{\widetilde{L}_0,\widetilde{L}_i}\big(\widetilde{\ell}(t)\big)$, by part (b) of Lemma 2.9. Finally, the signature of the restriction of B_t to $\pi(S)$ is independent of t, by (e) of Lemma 2.9. This concludes the proof.

□

An interesting immediate consequence of Theorem 2.11 is the following:

Corollary 2.12. *Let S be an isotropic subspace of V, and let $\ell : [a,b] \to \Lambda$ be a continuous curve such that $S \subset \ell(t)$ for all $t \in [a,b]$. If L_0 and L_0' are two Lagrangians such that:*

$$(L_0 \cap S^\perp) + S = (L_0' \cap S^\perp) + S, \tag{2.13}$$

then $\mu_{L_0}(\ell) = \mu_{L_0'}(\ell)$.

Proof: Simply note that if (2.13) holds, then $q(L_0 \cap S^\perp) = q(L_0' \cap S^\perp)$.

□

Another immediate consequence[1] of Theorem 2.11 and Lemma 2.2 is the following:

Corollary 2.13. *Let $L_0, L_1, L_2 \in \Lambda$ any three Lagrangians, and let S be an isotropic subspace contained in the sum $(L_0 \cap L_1) + (L_0 \cap L_2) + (L_1 \cap L_2)$. Then:*

$$\tau(L_0, L_1, L_2) = \tau(\widetilde{L}_0, \widetilde{L}_1, \widetilde{L}_2),$$

where $\widetilde{L}_i = q(L_i \cap S^\perp)$ and $q : S^\perp \to S^\perp/S$ is the quotient map.

Using formula (2.2) we have an analogous result for the Hörmander index:

Corollary 2.14. *Let $L_0, L_1, L_2, L_3 \in \Lambda$ any four Lagrangians, and let S be an isotropic subspace contained in $(L_0 \cap L_1)$. Then:*

$$\mathfrak{q}(L_0, L_1; L_2, L_3) = \mathfrak{q}(\widetilde{L}_0, \widetilde{L}_1; \widetilde{L}_2, \widetilde{L}_3),$$

where $\widetilde{L}_i = q(L_i \cap S^\perp)$ and $q : S^\perp \to S^\perp/S$ is the quotient map.

[1] Corollary 2.13 might have a direct proof using the definition of Kashiwara index in terms of index of a quadratic form, as in [6].

3 Comparison of Maslov indexes

Throughout this section, we will fix a symplectic space (V,ω) with $\dim(V) = 2n$. Let us consider a continuous Lagrangian path $\ell : [a,b] \to \Lambda$ and two distinct fixed Lagrangians $L_0, L_0' \in \Lambda$, and we want to compare the Maslov indexes $\mu_{L_0}(\ell)$ and $\mu_{L_0'}(\ell)$. Let us start with a preliminary result that has some interest of its own.

Lemma 3.1. *Assume that $\ell : [a,b] \to \Lambda$ is a continuous curve such that there exists some $L \in \Lambda$ with $\ell(t) \cap L = \{0\}$ for all $t \in [a,b]$. Then for any other Lagrangian $L_0 \in \Lambda$, $|\mu_{L_0}(\ell)| \le n$.*

Proof: We claim that there exists $L_1 \in \Lambda^0(L_0)$ such that $\ell(t) \cap L_1 = \{0\}$ for all $t \in [a,b]$. To prove this assertion, consider the set:

$$\Lambda^2_* = \big\{(\alpha,\beta) \in \Lambda \times \Lambda : \alpha \cap \beta = \{0\}\big\},$$

which is an open (and dense) subset of $\Lambda \times \Lambda$. The compact set $\ell\big([a,b]\big) \times \{L\}$ is contained in Λ^2_*, and thus there exists an open neighborhood U of L in Λ such that $\ell\big([a,b]\big) \times U$ is also contained in Λ^2_*. Since $\Lambda^0(L_0)$ is an open dense subset of Λ, by Baire's theorem the intersection $V \cap \Lambda^0(L_0)$ is non empty. The desired Lagrangian L_1 is any element of this intersection.

Now, to compute $\mu_{L_0}(\ell)$, we use the chart φ_{L_0,L_1}, whose domain contains entirely the support of the curve ℓ. We thus obtain:

$$|\mu_{L_0}(\ell)| = \Big|\tfrac{1}{2}\mathrm{sign}\big[\varphi_{L_0,L_1}\big(\ell(b)\big)\big] - \tfrac{1}{2}\mathrm{sign}\big[\varphi_{L_0,L_1}\big(\ell(a)\big)\big]\Big| \le n,$$

which concludes the proof.

$\square$

Lemma 3.2. *Assume that $\ell(a) \cap L_0' = \ell(b) \cap L_0' = \{0\}$. Then:*

$$\big|\mu_{L_0}(\ell) - \mu_{L_0'}(\ell)\big| \le n. \tag{3.1}$$

Proof: Choose any continuous path $\overline{\ell} : [b, c] \to \Lambda^0(L_0')$ with $\overline{\ell}(b) = \ell(b)$ and $\overline{\ell}(c) = \ell(a)$. Such choice is possible, because the open set $\Lambda^0(L_0')$ is diffeomorphic to a vector space, thus it is arc-connected. Denote by $\widetilde{\ell}$ the concatenation $\ell \diamond \overline{\ell} : [a, c] \to \Lambda$, which is a closed loop, and so:

$$\mu_{L_0}(\ell) + \mu_{L_0}(\bar{\ell}) = \mu_{L_0}(\widetilde{\ell}) = \mu_{L_0'}(\widetilde{\ell}) = \mu_{L_0'}(\ell) + \mu_{L_0'}(\bar{\ell}).$$

Since $\overline{\ell}(t) \in \Lambda^0(L_0')$ for all t, it follows that $\mu_{L_0'}(\bar{\ell}) = 0$, hence:

$$\left|\mu_{L_0}(\ell) - \mu_{L_0'}(\ell)\right| = |\mu_{L_0}(\bar{\ell})|.$$

Since $\overline{\ell}\big([b, c]\big) \subset \Lambda^0(L_0')$, by Lemma 3.1 $|\mu_{L_0}(\bar{\ell})| \leq n$, which concludes the proof.

□

We want to find an estimate of the difference $\left|\mu_{L_0}(\ell) - \mu_{L_0'}(\ell)\right|$ for an arbitrary curve ℓ, and the following technical result will be needed:

Lemma 3.3. *Given three Lagrangians $\ell_0, L_0, L_1 \in \Lambda$, there exists a continuous (in fact, smooth) curve $\ell_\pm : [a, b] \to \Lambda$ with the following properties:*

(a) $\ell_\pm(a) = \ell_0$*;*

(b) $\ell_\pm(t) \in \Lambda^0(L_0) \cap \Lambda^0(L_1)$ *for all* $t \in \,]a, b]$*;*

(c) $\mu_{L_0}(\ell_\pm) = \mu_{L_1}(\ell_\pm)$ *if* $\dim(L_0 \cap \ell_0) \equiv \dim(L_1 \cap \ell_0) \mod 2$*;*

(d) $\mu_{L_0}(\ell_\pm) - \mu_{L_1}(\ell_\pm) = \pm\frac{1}{2}$ *if* $\dim(L_0 \cap \ell_0) \not\equiv \dim(L_1 \cap \ell_0) \mod 2$*.*

Proof: One can find a symmetric bilinear form $B_\pm$ on ℓ_0 such that the restrictions of $B_\pm$ to $\ell_0 \cap L_0$ and to $\ell_0 \cap L_1$, denoted respectively by $B_\pm^0$ and $B_\pm^1$, are both nondegenerate. Moreover, if $\dim(\ell_0 \cap L_0)$ and $\dim(\ell_0 \cap L_1)$ have the same parity, $B_\pm$ can be chosen so that the signatures of $B_\pm^0$ and $B_\pm^1$ coincide. If $\dim(\ell_0 \cap L_0)$ and $\dim(\ell_0 \cap L_1)$ have different parities, then B can be chosen in such a way that the difference $\mathrm{sign}(B_\pm^0) - \mathrm{sign}(B_\pm^1) = \pm 1$. Let now $\ell_\pm : [a, b] \to \Lambda$ be any smooth curve such that $\ell_\pm(a) = \ell_0$ and $\ell_\pm'(a) = B$. Since $B_\pm^0$ and $B_\pm^1$ are nondegenerate, then for $b - a > 0$ sufficiently small,

$\ell_\pm(t) \in \Lambda^0(L_0) \cap \Lambda^0(L_1)$ for all $t \in\,]a, b]$. Moreover, the Maslov index $\mu_{L_i}(\ell_\pm)$ of such a curve is given by half of the signature of the restriction of $\ell'_\pm(a)$ to $\ell_\pm(a) \cap L_i = \ell_0 \cap L_i$, for $i = 0, 1$. The conclusion follows easily.

$\square$

Corollary 3.4. *Given any continuous curve $\ell : [a, b] \to \Lambda$ and any pair $L_0, L_1 \in \Lambda$ of Lagrangians, then:*

$$\big|\mu_{L_0}(\ell) - \mu_{L_1}(\ell)\big| \le n + \frac{1}{2}. \tag{3.2}$$

If:

$$\dim\big(L_0 \cap \ell(a)\big) + \dim\big(L_1 \cap \ell(a)\big) + \dim\big(L_0 \cap \ell(b)\big) + \dim\big(L_1 \cap \ell(b)\big) \in 2\mathbb{Z}, \tag{3.3}$$

then:

$$\big|\mu_{L_0}(\ell) - \mu_{L_1}(\ell)\big| \le n. \tag{3.4}$$

Proof: By Lemma 3.3 we can find continuous curves $\ell_i : [a_i, b_i] \to \Lambda$, $i = 1, 2$, such that $\ell_1(a_1) \in \Lambda^0(L_0) \cap \Lambda^0(L_1)$, $\ell_1(b_1) = \ell(a)$, $\ell_2(a_2) = \ell(b)$, $\ell_2(b_2) \in \Lambda^0(L_0) \cap \Lambda^0(L_1)$ and with

$$\mu_{L_0}(\ell_i) - \mu_{L_1}(\ell_i) \in \big\{0, \pm\tfrac{1}{2}\big\}, \quad i = 1, 2, \tag{3.5}$$

depending on the parity of the dimensions of $\ell(a) \cap L_0$, $\ell(a) \cap L_1$, $\ell(b) \cap L_0$, and $\ell(b) \cap L_1$. More precisely, using parts (c) and (d) of Lemma 3.3 one sees that the quantity:

$$\mu_{L_0}(\ell_1) - \mu_{L_1}(\ell_1) - \mu_{L_0}(\ell_2) + \mu_{L_1}(\ell_2)$$

can be made equal to 0 if the numbers

$$\dim\big(\ell(a) \cap L_0\big) - \dim\big(\ell(a) \cap L_1\big) \quad \text{and} \quad \dim\big(\ell(b) \cap L_0\big) - \dim\big(\ell(b) \cap L_1\big)$$

have the same parity, i.e., if

$$\dim\big(L_0 \cap \ell(a)\big) + \dim\big(L_1 \cap \ell(a)\big) + \dim\big(L_0 \cap \ell(b)\big) + \dim\big(L_1 \cap \ell(b)\big)$$

is an even number, and equal to $\pm\frac{1}{2}$ otherwise.

Now, choose any continuous curve $\overline{\ell} : [c, d] \to \Lambda^0(L_1)$ such that $\overline{\ell}(c) = \ell_2(b_2)$ and $\overline{\ell}(d) = \ell_1(a_1)$, and thus, by Lemma 3.1:

$$|\mu_{L_0}(\overline{\ell})| \le n, \quad \mu_{L_1}(\overline{\ell}) = 0. \tag{3.6}$$

Now consider the closed loop $\widetilde{\ell}$ given by the concatenation $\ell \diamond \ell_2 \diamond \overline{\ell} \diamond \ell_1$, which gives:

$$\begin{aligned}\mu_{L_0}(\ell) + \mu_{L_0}(\ell_2) + \mu_{L_0}(\overline{\ell}) + \mu_{L_0}(\ell_1) &= \mu_{L_0}(\widetilde{\ell}) \\ &= \mu_{L_1}(\widetilde{\ell}) = \mu_{L_1}(\ell) + \mu_{L_1}(\ell_2) + \mu_{L_1}(\overline{\ell}) + \mu_{L_1}(\ell_1),\end{aligned}$$

and so, using (3.6):

$$\begin{aligned}&\big|\mu_{L_0}(\ell) - \mu_{L_1}(\ell)\big| \\ &\quad \le \big|\mu_{L_1}(\ell_1) - \mu_{L_0}(\ell_1) + \mu_{L_1}(\ell_2) - \mu_{L_0}(\ell_2)\big| + \big|\mu_{L_0}(\overline{\ell})\big| + \big|\mu_{L_1}(\overline{\ell})\big| \\ &\qquad \le n + \big|\mu_{L_1}(\ell_1) - \mu_{L_0}(\ell_1) + \mu_{L_1}(\ell_2) - \mu_{L_0}(\ell_2)\big|.\end{aligned}$$

The conclusion follows easily.

□

We will now establish a more precise inequality for the difference $\mu_{L_0}(\ell) - \mu_{L_0'}(\ell)$ using a symplectic reduction.

Proposition 3.5. *Given any continuous curve $\ell : [a, b] \to \Lambda$ and any pair $L_0, L_1 \in \Lambda$ of Lagrangians, then:*

$$\big|\mu_{L_0}(\ell) - \mu_{L_1}(\ell)\big| \le n - \dim(L_0 \cap L_1) + \epsilon, \tag{3.7}$$

where $\epsilon = 0$ if (3.3) *holds, and $\epsilon = \frac{1}{2}$ otherwise.*

Proof: Consider the isotropic space $S = L_0 \cap L_1$; denote by $q : S^\perp \to S^\perp/S$ the quotient map and, for $L \in \Lambda$, by $\widetilde{L}$ the Lagrangian in the isotropic reduction $S^\perp/S$ given by $q(L \cap S^\perp)$. By Corollary 2.14:

$$\mu_{L_0}(\ell) - \mu_{L_1}(\ell) = \mathfrak{q}\big(L_0, L_1; \ell(a), \ell(b)\big) = \mathfrak{q}\big(\widetilde{L}_0, \widetilde{L}_1; \widetilde{\ell(a)}, \widetilde{\ell(b)}\big).$$

By Corollary 3.4:

$$\left|\mathfrak{q}\big(\widetilde{L}_0,\widetilde{L}_1;\widetilde{\ell(a)},\widetilde{\ell(b)}\big)\right| \le \tfrac{1}{2}\dim(S^{\perp}/S)+\epsilon = n-\dim(L_0\cap L_1)+\epsilon,$$

where $\epsilon = 0$ if

$$\dim\big(\widetilde{L}_0\cap\widetilde{\ell(a)}\big)+\dim\big(\widetilde{L}_1\cap\widetilde{\ell(a)}\big)+\dim\big(\widetilde{L}_0\cap\widetilde{\ell(b)}\big)+\dim\big(\widetilde{L}_1\cap\widetilde{\ell(b)}\big) \qquad (3.8)$$

is an even number and $\epsilon = \frac{1}{2}$ otherwise. Since $S = L_0 \cap L_1$, it is easily computed:

$$\dim(\widetilde{L}\cap\widetilde{L}_i) = \dim(L\cap L_i) - \dim(L\cap L_0\cap L_1)$$

for all $L \in \Lambda$ and $i = 0, 1$. It follows easily that (3.8) and the integer in (3.3) have the same parity, which concludes the proof.

□

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