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THE EULER CLASS AND THE VOLUME FUNCTIONAL OF VECTOR FIELDS

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Abstract

A vector field X on a riemannian manifold M determines a submanifold in the tangent bundle. The volume of X is the volume of this submanifold for the induced Sasaki metric. When M is compact, the volume is well defined and, usually, this functional is studied for unit fields. Parallel vector fields are trivial minima of this functional.

For odd-dimensional manifolds, we obtain an explicit result showing how the topology of a vector field with constant length influences its volume. We apply this result to the case of vector fields that define riemannian foliations with all leaves compact.

1 Introduction

Let (M^n, g) be an n-dimensional closed riemannian manifold and X a smooth vector field on M. The metric g induces a natural metric on the tangent bundle TM, usually called the sasakian metric. The volume of X is defined as the volume of the section $X : M \to TM$, see [GZ]. An expression of the volume of X in terms of the Levi-Civita connection ∇ of (M, g) is:

$$vol(X) = \int_M \left(1 + \sum_{a=1}^{n} \|\nabla_{e_a} X\|^2 + \sum_{a_1<a_2} \left\|\nabla_{e_{a_1}} X \wedge \nabla_{e_{a_2}} X\right\|^2 + \cdots + \right.$$
$$\left. + \cdots + \|\nabla_{e_1} X \wedge \ldots \wedge \nabla_{e_n} X\|^2 \right)^{\frac{1}{2}} \nu \quad (1)$$

where ν is the volume form of M and $\{e_a\}_{a=1}^n$ is an orthonormal local frame. Note that for any vector field $vol(X) \geqslant vol(M)$. The zero section has minimum volume (for the moment, X is merely a smooth flow). From a geometric viewpoint, the first natural restriction would be to consider the functional simply on unit vector fields. Possible, vector fields with constant length are the following step. Note that this requirement is also a restriction for M (for example, the Euler characteristic of M must be zero).

For the case of unit vector fields, only parallel fields on M attain the trivial minimum for volume. Unit parallel vector fields are rare in manifolds of dimensional higher than 1. In fact, the existence of parallel fields implies that locally the manifold is a riemannian product. For example, odd-dimensional spheres of any radius (except $\mathbb{S}^1$) admit no unit parallel vector field.

In $\mathbb{S}^3$ we know from [GZ] that Hopf vector fields V_H (The unit flow tangent to the classical Hopf fibration) and no others, are the minima of the volume (among globally defined unit vector fields).

We prove the following theorem relating the geometry and topology of arbitrary vector fields with constant length.

Theorem 1.1 *Let M be a compact riemannian manifold of dimension $2k+1$. Let $r>0$ and X a vector field over M such that $\|X\| = \frac{1}{\sqrt{r}}$. Denote by θ_{2k+1} the unit dual form to X and by $\mathcal{R}^2$ the sum $\sum_{1\leqslant i,j,k,l\leqslant 2k} R^2_{ijkl}$, where R_{ijkl} are the components of the curvature tensor of M. If $r \geqslant \frac{1}{2\sqrt{k(2k-1)}}\mathcal{R}$, then*

$$vol(X) \geqslant \frac{(4\pi)^k k!}{r^k (2k)!}\left\| \int_M \chi(X^{\perp}) \wedge \theta_{2k+1} \right\| \tag{2}$$

Where $\chi(X^{\perp})$ is the Euler form of the orthogonal subbundle to X.

A first reading of this theorem suggests that the volume of a vector field of constant length is greater than or equal to some constant times the integral of the Lipschitz-Killing curvature of the distribution complementary to the field.

From now on we adopt the same notation as in [BC]. We think useful to make some remarks about the theorem before proving it.

2 Examples

a) If $M = S^{2k+1}$ is the standard unit sphere, computation shows that

$$-\Omega_{ij} = \theta_i \wedge \theta_j.$$

This equation expresses the fact that the sectional curvature of the unit sphere is identically equal to +1. Furthermore, by definition, choosing an oriented, orthonormal basis $\theta_1, \ldots, \theta_k$ for the sections of $X^\perp$,

$$(-1)^n Pf(\tilde{\Omega}) = Pf(\theta_i \wedge \theta_j) = (1.3.5.\cdots.(2k-1))\theta_1 \wedge \ldots \wedge \theta_{2k}.$$

so,

$$\frac{(4k)^k k!}{(2k)! r^k} \left\| \int_M \chi(X^\perp) \wedge \theta_{2k+1} \right\| = \frac{(2)^k k!}{(2k)! r^k} \left\| \int_{S^{2k+1}} Pf(\tilde{\Omega}) \wedge \theta_{2k+1} \right\| =$$
$$= \frac{(2)^k k!}{(2k)! r^k} (1.3.5.\cdots.(2k-1)) \int_{S^{2k+1}} dS^{2k+1} =$$
$$= \frac{(2)^k k!}{(2k)! r^k} \cdot \frac{(2k)!}{2^k k!} \int_{S^{2k+1}} dS^{2k+1} = \frac{1}{r^k} vol(S^{2k+1}).$$

b) Supposing $X^\perp$ to be an integrable normal bundle, we have

$$\int_M \chi(X^\perp) \wedge \theta_{2k+1} = \int_M \kappa_{x^\perp(p)} \nu,$$

where $\kappa_{X^\perp} : M \to \mathbb{R}$ is the Lipschitz-Killing curvature of the leave through $p \in M$, of the foliation $X^\perp$; Recall that the definition of Lipschitz-Killing curvature of a manifold M at a point $p \in M$ is $\kappa(p) = \frac{1}{(2\pi)^n} Pf(\Omega_{ij})$, where ν is the volume element of M and $\Omega = (\Omega_{ij})$ is a $2n$ globally defined differential form on M.

c) $S^3 \times S^2$

The integral of Lipschitz-Killing of the leaves may be, in some cases expressed only in terms of the metric invariants of the manifold (see [BLR]). This computation, in the non constant curvature case is, as far as we know, undone. The three dimensional case is easier to compute and reads:

Proposition 2.1 *If M^3 is a closed 3-dimensional riemannian manifold, $\mathfrak{F}$ a transversely oriented codimension one foliation of M^3. Then*

$$\int_M \kappa_{\mathfrak{F}} = 3\int_M \kappa - \int MRic(\mathfrak{F}^{\perp})$$

where $\kappa(p)$ is the scalar curvature of M at p and $Ric(\mathfrak{F}^{\perp})$ is the Ricci curvature in the normal direction to $\mathfrak{F}$, and $\kappa(p)$ is the gaussian curvature of $\mathfrak{F}$ at p;

A detailed proof for a similar result can be found in [B].

d) In totally symmetric spaces, the tensor R is locally constant, and, so is its lengths. Ins some particular spaces like product of round spheres, complex projective spaces, or some classical Lie groups, one can make explicit the norm $\|R\|$ of the curvature tensor which appear in the statement of the main theorem.

e) Finally, notice that the main result of this paper is a natural extension of the one appearing in [BC] which states:

Theorem 2.2 *Let M be a compact riemannian manifold of dimension 5. Let $r > 0$ and X a vector field over M such that $\|X\| = \frac{1}{\sqrt{r}}$. Denote by θ_5 the unit dual form to X and by $\mathcal{R}^2$ the sum $\sum_{1\leqslant i,j,k,l\leqslant 4} R^2_{ijkl}$, where R_{ijkl} are the components of the curvature tensor of M. If $r \geqslant \frac{1}{2\sqrt{6}}\mathcal{R}$, then*

$$vol(X) \geqslant \frac{4\pi^2}{3r^2}\left\|\int_M \chi(X^{\perp}) \wedge \theta_5\right\| \qquad (3)$$

Where $\chi(X^{\perp})$ is the Euler form of the orthogonal subbundle to X.

3 Proof of the theorem.

Associated with Y we have the unit vector field given by $X = \sqrt{r}Y$. Consider a local frame $\{e_1, \ldots, e_{2k+1} = X\}$ adapted to X. Disregarding the terms involving the acceleration of X (that is, the terms where $\nabla_X X$ appear):

$$vol(Y) \geqslant \frac{1}{r^k}\int_M \Bigg(r^{2k} + r^{2k-1}\sum_a \|\nabla_{e_a}X\|^2 + r^{2k-2}\sum_{a_1<a_2} \left\|\nabla_{e_{a_1}}X\wedge\nabla_{e_{a_2}}X\right\|^2 + \\ + r^{2k-3}\sum_{a_1<a_2<a_3}\left\|\nabla_{e_{a_1}}X\wedge\nabla_{e_{a_2}}X\nabla_{e_{a_3}}X\right\|^2 + \ldots + \\ + \ldots + \sum_{a_1<\ldots<a_{2k}} \|\nabla_{e_1}X\wedge\ldots\wedge\nabla_{e_{2k}}X\|^2\Bigg)^{\frac{1}{2}}\nu \tag{4}$$

where all the indices run form 1 to $2k$.

The first sum of (4) is merely the sum of squares of all the entries of the matrix associated with the second fundamental form $\mathcal{H}$. That is,

$$\sum_{a=1}^{2k}\|\nabla_{e_a}X\|^2 = \sum_{a,b=1}^{2k} h_{ab}^2$$

The second and third sums of (4) are respectively the sum of the squares of all the 2×2 and 3×3 minors of $\mathcal{H}$. Finally, the last sum in (4) is simply the square of the determinant of $\mathcal{H}$. Like in [BC], we denote by $(\Delta_i)^2$ the sum of the squares of all the $i\times i$ minors of $\mathcal{H}$. With this, we can rewrite (4) as:

$$vol(Y) \geqslant \frac{1}{r^k}\int_M \left(r^{2k} + r^{2k-1}(\Delta_1)^2 + r^{2k-2}(\Delta_2)^2 + \ldots + r(\Delta_{2k-1})^2 + (\Delta_{2k})^2\right)^{\frac{1}{2}}\nu \tag{5}$$

Let us consider the $2k\times 2k$ symmetric matrix $A = (g(\nabla_{e_a}X, \nabla_{e_b}X))_{ab}$, The elementary symmetric functions of A, $\sigma_i(A)$, are exactly the terms Δ_i^2 :

$$det(Id + tA) = \sum_{i=0}^{2k}\sigma_i(A)t^i = \sum_{i=0}^{2k}(\Delta_i)^2 t^i$$

where $\Delta_0^2 = \sigma_0(A) = 1$. The normalized symmetric functions satisfy the following properties, see [HLP] and [BCN]:

For j even and $s = 0, 1, \ldots, \frac{j}{2}$,

$$\sigma_j^2 \geqslant \frac{\binom{2k}{j}^2}{\binom{2k}{j-2s}\binom{2k}{j+2s}}\sigma_{j-2s}\sigma_{j+2s}.$$

and for j odd and $s = 0, \dots, \frac{j-1}{2}$,

$$\sigma_j^2 \geqslant \frac{\binom{2k}{j}^2}{\binom{2k}{j-2s-1}\binom{2k}{j+2s+1}}\sigma_{j-2s-1}\sigma_{j+2s+1}.$$

Then, for j even and $s = 0, \dots, \frac{j}{2}$,

$$\Delta_j^4 \geqslant \frac{\binom{2k}{j}^2}{\binom{2k}{j-2s}\binom{2k}{j+2s}}\Delta_{j-2s}^2\Delta_{j+2s}^2,$$

but since all the Δ_i are positive, we have:

$$\Delta_j^2 \geqslant \frac{\binom{2k}{j}}{\sqrt{\binom{2k}{j-2s}\binom{2k}{j+2s}}}\Delta_{j-2s}\Delta_{j+2s}. \tag{6}$$

For the other case,

$$\Delta_j^2 \geqslant \frac{\binom{2k}{j}}{\sqrt{\binom{2k}{j-2s-1}\binom{2k}{j+2s+1}}}\Delta_{j-2s-1}\Delta_{j+2s+1}. \tag{7}$$

Now we need of the following preliminary lemma.

Lemma 3.1 *with the same notations adopted until now, we have that*

1. *(i)For each $j = 1, \dots, 2k$,*

$$\Delta_j^2 \geqslant \sum_{i=0}^{j} \frac{\binom{k}{i}\binom{k}{j-i}}{\sqrt{\binom{2k}{2i}\binom{2k}{2j-2i}}}\Delta_{2i}\Delta_{2j-2i} \tag{8}$$

2. *(ii)For each $k \geqslant 1$ e $0 \leqslant i \leqslant k$, (i and k integers),*

$$\frac{(2k)!}{(2i)!} \geqslant \frac{(4\binom{2k}{2})^{k-i}}{(2k-2i)!} \tag{9}$$

Proof: Part (i), is proven in [BCN].

In order to show (ii), we consider the function $f:[0,\infty)\times[0,\infty)\longrightarrow\mathbb{R}$ given by

$$f(k,x)=\frac{(2k)!}{\Gamma(2x+1)}-\frac{(4\binom{2k}{2})^{k-x}}{\Gamma(2k-2x+1)}$$

where $\Gamma(x)$ is the restriction of the Euler gamma function to $\mathbb{R}$. Computing its derivative we can see that

$$\frac{\partial f}{\partial x}=\frac{(2k)!}{2\Gamma(2x+1)}+\frac{(4\binom{2k}{2})^{k-x}}{(\Gamma(2k-2x+1))^2}\Big[\Gamma(2k-2x+1)\ln\Big(4\binom{2k}{2}\Big)+ \\ +4\frac{\partial}{\partial x}\Gamma(2k+2x+1)\Big]>0,$$

Notice that f is an increasing function on the second variable. It is then sufficient to prove (ii) for $x=0\ \forall k\geqslant 1$.

$$((2k)!)^2\geqslant\Big(4\binom{2k}{2}\Big)^k \tag{10}$$

The proof follows by a induction argument: For $k=1$, (10) verifies. Suppose that (10) is true, we need to show that

$$((2k+2)!)^2\geqslant\Big(4\binom{2k+2}{2}\Big)^{k+1}. \tag{11}$$

Because $\binom{2k+2}{2}=\frac{(2k+2)(2k+1)}{2k(2k-1)}\binom{2k}{2}$, follows that (11) is equivalent to

$$(2k+2)^2(2k+1)^2((2k)!)^2\geqslant 4\binom{2k+2}{2}\Big[\frac{(2k+2)(2k+1)}{2k(2k-1)}\Big]^k 4^k\binom{2k}{2}^k \tag{12}$$

with $\binom{2k+2}{2}=(k+1)(2k+1)$, and the induction hypothesis, it is clear that it is sufficient for the proof of (12), to show that

$$4(k+1)^2(2k+1)^2\geqslant 4(k+1)(2k+1)\Big[(1+\frac{1}{k})(1+\frac{2}{2k-1})\Big]^k,$$

or, equivalently,

$$(k+1)(2k+1)\geqslant\Big(1+\frac{1}{k}\Big)^k\Big(1+\frac{2}{2k-1}\Big)^k. \tag{13}$$

notice that, $a_k = (1+\frac{1}{k})^k$ is a bounded increasing positive real sequence, $a_k < 3$. Then $(1+\frac{2}{2k-1})^k < 3\sqrt{2} < 6$, and this implies that $b_k = (1+\frac{1}{k})^k(1+\frac{2}{2k-1})^k$ is a bounded increasing real sequence, with $b_k < 18$. Also, $c_k = (k+1)(2k+1)$ is strictly increasing, $c_k \geqslant 28$ for $k \geqslant 4$, and, because this (13) is true for $k \geqslant 4$. By direct computation, (13) is true for $1 \leqslant k \leqslant 3$, and the lemma is proven.

□

By (5), (8) and the lemma(3.1)

$$\begin{aligned} vol(Y) &\geqslant \frac{1}{r^k}\int_M \Big(\sum_{j=0}^{2k}\Delta_j^2 r^{2k-j}\Big)^{\frac{1}{2}}\nu \geqslant \\ &\geqslant \frac{1}{r^k}\int_M \left(\sum_{j=0}^{2k}\sum_{i=0}^{j}\frac{\binom{k}{i}\binom{k}{j-i}}{\sqrt{\binom{2k}{2i}\binom{2k}{2j-2i}}}\Delta_{2i}\Delta_{2j-2i}r^{2k-j}\right)^{\frac{1}{2}}\nu \geqslant \\ &\geqslant \frac{1}{r^k}\int_M \left(\left[\sum_{i=0}^{k}\frac{\binom{k}{i}}{\sqrt{\binom{2k}{2i}}}\Delta_{2i}r^{k-i}\right]^2\right)^{\frac{1}{2}}\nu \geqslant \\ &\geqslant \frac{1}{r^k}\int_M \left\|\frac{1}{(2k)!}\sum_{i=0}^{k}(2i)!\binom{k}{i}\mathcal{R}^{k-i}\Delta_{2i}\right\|\nu. \end{aligned} \tag{14}$$

The last integral is related to the Pfaffian of $X^{\perp}$ in the following way (see [BC])

$$\begin{aligned} &2^k(k)!Pf(\widetilde{\Omega}) = \\ &= \sum_{\tau\in\mathfrak{S}_{2k}}\epsilon(\tau)\Big(\Omega_{\tau(1)\tau(2)}+\omega_{\tau(1)2k+1}\wedge\omega_{\tau(2)2k+1}\Big)\wedge\Big(\Omega_{\tau(3)\tau(4)}+\omega_{\tau(3)2k+1}\wedge\omega_{\tau(4)2k+1}\Big) \\ &\qquad \wedge\ldots\wedge\Big(\Omega_{\tau(2k-1)\tau(2k)}+\omega_{\tau(2k-1)2k+1}\wedge\omega_{\tau(2k)2k+1}\Big). \end{aligned}$$

Then,

$$\begin{aligned} &\sum_{\tau\in\mathfrak{S}_{2k}}\epsilon(\tau)\Omega_{\tau(1)\tau(2)}\wedge\ldots\wedge\Omega_{\tau(2k-(2k-i))\tau(2k-2i)}\wedge\omega_{\tau(2k-2i+1)2k+1}\wedge\ldots\wedge\omega_{\tau(2k)2k+1} \\ &= 2^{(k-i)}(2i)!\sum_{a_j<a_{j+1}}\varepsilon(a)\Omega_{a_1a_2}\wedge\ldots\wedge\Omega_{a_{2k-(2i+1)}a_{2k-2i}}\wedge\omega_{a_{2k-2i+1}2k+1}\wedge\ldots\wedge\omega_{a_{2k}2k+1} \end{aligned}$$

where each a_i to run from 1 to $2k$ and, if $i \neq j$, the pair (a_i, a_{i+1}) is not ordered with respect to (a_j, a_{j+1}).
Denoting by $\binom{k}{i}P_i$ the i-th term of the expression for the Pfaffian and applying to $\{e_1, \dots, e_{2k}\}$, we have, applying the Cauchy-Schwartz inequality we obtain:

$$\begin{aligned} P_i(e_1, \dots, e_{2k}) &\leqslant 2^{(k-i)}(2i)!\Big(\sum_{\substack{a_1<a_2\\ b_1<b_2}} R^2_{a_1a_2b_1b_2}\Big)^{\frac{1}{2}} \cdot \Big(\sum_{\substack{a_3<a_4\\ b_3<b_4}} R^2_{a_3a_4b_3b_4}\Big)^{\frac{1}{2}} \cdot \\ &\cdot \dots \cdot \Big(\sum_{\substack{a_{2k-(2i+1)}<a_{2k-2i}\\ b_{2k-(2i+1)}<b_{2k-2i}}} R^2_{a_{2k-(2i+1)}a_{2k-2i}b_{2k-(2i+1)}b_{2k-2i}}\Big)^{\frac{1}{2}} \cdot (\Delta^2_{2i})^{\frac{1}{2}} \\ &\leqslant 2^{(k-i)}(2i)! \underbrace{\big(\frac{1}{4}\mathcal{R}^2\big)^{\frac{1}{2}} \cdot \dots \cdot \big(\frac{1}{4}\mathcal{R}^2\big)^{\frac{1}{2}}}_{(k-i)-\text{terms}} \cdot (\Delta^2_{2i})^{\frac{1}{2}} \\ &\leqslant (2i)!\mathcal{R}^{(k-i)}\Delta_{2i}; \end{aligned} \tag{15}$$

therefore, we have:

$$Pf(\widetilde{\Omega})(e_1, \dots, e_{2k}) \leqslant \frac{(2k!)}{2^k k!}\left(\frac{1}{(2k)!}\sum_{i=0}^{k}(2i)!\binom{k}{i}\mathcal{R}^{(k-i)}\Delta_{2i}\right). \tag{16}$$

Finally, by this inequality, by (14) and by the definition of the Pfaffian of the curvature (in terms of the Euler class)

$$vol(Y) \geqslant \frac{2^k k!}{(2k)!r^k}\left\|\int_M Pf(\widetilde{\Omega}) \wedge \theta_{2k+1}\right\| = \frac{(4\pi)^k k!}{(2k)!r^k}\left\|\int_M \chi(X^{\perp}) \wedge \theta_{2k+1}\right\|$$

and this ends the proof of the theorem.

□

Like in [BC], is valid the following

Remark 3.2 *: In view of the proof, the inequality of Theorem 2.2 will be sharp int the following situation:*

1. *(1)The flow of the field is geodesic;*

2. *(2)All the eigenvalues of the symmetric matrix A are equal;*

3. *(3) The curvature of M satisfy several relations, such proportionality:* $R_{a_1a_2b_1b_2} = \lambda R_{a_3a_4b_3b_4}$*, where* $a_n < a_{n+1}$ *and* $b_m < b_{m+1}$ *(por 15);*

4. *(4)*
$$R_{a_1a_2b_1b_2} = \mu \det \begin{pmatrix} h_{a_3b_3} & \cdots & h_{a_3b_{2i+2}} \\ \vdots & \ddots & \vdots \\ h_{a_{2i+2}b_3} & \cdots & h_{a_{2i+2}b_{2i+2}} \end{pmatrix}$$

In spaces of constant sectional curvature the conditions hold when X is geodesic and $X^\perp$ is integrable with umbilical leaves. It would be very interesting to know if the inequality can be attained in other manifolds for some specific field.

3.1 A topological corollary

Like in [BC], we observe which the minorization of the Theorem 2.2 involves the Euler class of the bundle $X^\perp$ and the dual form to the unit field X. When this form is closed then $X^\perp$ will be a riemannian foliation. This means that the flow of Y is geodesic and the distribution $X^\perp$ is integrable.

In this situation, if the foliation defined by $X^\perp$ has a compact leaf then all the leaves are compact and M is a fiver bundle over $\mathbb{S}^1$ (a circle of length L) the fiber being the compact leaves of the foliation (see [Mo]). We are now in a position to state the following consequence.

Corollary 3.3 *Let M be a closed riemannian manifold of dimension $2k+1$ and X a vector field of constant length $\frac{1}{\sqrt{r}}$ such that its unit dual form is closed and the orthogonal foliation $X^\perp$ has a compact leaf. Thus, M fibers over a circle of length L. Let us assume that $r \geqslant \frac{1}{2\sqrt{k(2k-1)}}\mathcal{R}$; We thus have*

$$vol(X) \geqslant \frac{(4\pi)^k k! L}{r^k (2k)!} \left\| \chi(X^\perp) \right\|$$

where $\chi(X^\perp)$ *is the Euler characteristic of the leaves of* $X^\perp$*.*

Our corollary gives a partial description of how the topology of X influences its volume, at least when the unit dual form of X is closed and $X^\perp$ only has compact leaves. We also wish to remark that the constant length of Y, the factor $\frac{1}{\sqrt{r}}$ appears in the lower bound of Theorem 2.2. When the length of Y is small (i.e., flow is slow), then the lower bound of $vol(Y)$ is small.

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