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Separability of double cosets and conjugacy classes in 3-manifold groups

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Abstract

Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold of finite volume. We show that if H and K are abelian subgroups of Γ and $g \in \Gamma$, then the double coset HgK is separable in Γ. As a consequence we prove that if M is a closed, orientable, Haken 3-manifold and the fundamental group of every hyperbolic piece of the torus decomposition of M is conjugacy separable then so is the fundamental group of M. Invoking recent work of Agol and Wise, it follows that if M is a compact, orientable 3-manifold then $\pi_1(M)$ is conjugacy separable.

1 Introduction

The profinite topology on a group Γ is the coarsest topology in which every homomorphism from Γ to a finite group is continuous. When Γ is the fundamental group of a manifold M, the profinite topology on Γ encodes the finite-sheeted covering spaces of M, and as such is of great interest in the field of low-dimensional topology.

Definition 1.1. *If a subset X of Γ is closed in the profinite topology then X is called* separable. *Equivalently, for every $\gamma \in \Gamma - X$ there is a homomorphism ϕ from Γ to a finite group such that $\phi(\gamma) \notin \phi(X)$.*

1. *A group Γ is* residually finite *if the trivial subgroup is separable in Γ.*
2. *A group Γ is* conjugacy separable *if every conjugacy class in Γ is separable.*
3. *A group Γ is* subgroup separable *or* locally extended residually finite (LERF) *if every finitely generated subgroup of Γ is separable in Γ.*
4. *A group Γ is* double-coset separable *if for every pair H, K of finitely generated subgroups of Γ, and every $g \in \Gamma$ the double coset HgK is separable.*

Note that conjugacy separability and subgroup separability both imply residual finiteness, while double-coset separability implies subgroup separability.

Hempel showed, using Thurston's Geometrization Theorem, that the fundamental groups of Haken 3-manifolds are residually finite [17]. Scott proved that the fundamental groups of surfaces and Seifert-fibered 3-manifolds are subgroup separable [29], and asked whether the same holds for all 3-manifold groups. Burns, Karass and Solitar [9] answered this question in the negative by giving an example of a graph manifold with non-subgroup-separable fundamental group, but it follows from the main theorems of recent preprints of Agol [2] and Wise [34] that the fundamental groups of hyperbolic 3-manifolds are subgroup separable. In this paper, we are concerned with conjugacy separability and double-coset separability in 3-manifold groups.

1.1 Conjugacy separability

Our first theorem extends the results of [32], in which it was proved that the fundamental groups of graph manifolds are conjugacy separable.

Theorem 1.2. *Let M be a closed, orientable, Haken 3-manifold and let $N_1, \ldots, N_m$ be the pieces of the torus decomposition of M. If each $\pi_1(N_i)$ is conjugacy separable then $\pi_1(M)$ is conjugacy separable.*

This is an important step in the proof that the fundamental group of any compact, orientable 3-manifold is conjugacy separable, as we now explain. We follow the same broad strategy that Hempel used in his proof of residual finiteness, although there are more difficult technical obstacles to overcome.

Let M be a compact, orientable 3-manifold, possibly with boundary. We are interested in the question of whether or not $\pi_1(M)$ is conjugacy separable. If M has boundary then, cutting along compressing discs and appealing to the fact that a free product of conjugacy separable groups is conjugacy separable [31], we may assume that ∂M is incompressible. Let D be the double of M along ∂M. Then $\pi_1(M)$ is a retract of $\pi_1(D)$. In particular, a pair of elements is conjugate in $\pi_1(M)$ if and only if it is conjugate in $\pi_1(D)$, and it follows that if $\pi_1(D)$ is conjugacy separable then so is $\pi_1(M)$. In this way, we can reduce to the case in which M is orientable and closed. Passing to the pieces of the Kneser–Milnor decomposition, and appealing again to the fact that a free product of conjugacy separable groups is conjugacy separable, we can reduce further to the irreducible case.

The next step is to pass to the pieces of the torus decomposition of M, described by Jaco–Shalen [18] and Johannson [20]. This point is the heart of Hempel's argument. He proves a gluing theorem that reduces the residual finiteness of $\pi_1(M)$ to the residual finiteness of the fundamental groups of the pieces. In the context of conjugacy separability, Theorem 1.2 supplies the necessary gluing theorem.

This reduces the question of which 3-manifold groups are conjugacy separable to the geometric case. If M is a torus bundle over a circle then $\pi_1(M)$ is polycyclic, and so is conjugacy separable by a theorem of Remeslennikov [28]. Martino proved that the fundamental groups of Seifert-fibered 3-manifolds are conjugacy separable [24].

The hyperbolic case is much more difficult, but dramatic progress has been made recently. The crucial concept is the notion of a *special* group, introduced by Haglund and Wise [13][1]. Minasyan proved that special groups are conjugacy separable [26], while Chagas and the third author gave conditions under which conjugacy separability passes to finite extensions [10]. Taking these two results together, it follows that, if N is a hyperbolic 3-manifold and $\pi_1(N)$ is *virtually* special (that is, $\pi_1(N)$ has a special subgroup of finite index) then $\pi_1(N)$ is conjugacy separable.

Many hyperbolic 3-manifold and orbifold groups are known to be virtually special [7, 8, 10, 11], among them non-cocompact arithmetic and standard cocompact arithmetic lattices. Wise has announced a proof that the fundamental group of any hyperbolic 3-manifold containing an embedded geometrically finite surface is virtually special [33]; the heart of his proof is contained in [34]. Very recently, Agol has given a proof that the fundamental group of any closed hyperbolic 3-manifold group is virtually special [2]. His proof uses the work of [34], as well as Kahn and Markovic's resolution of the Surface Subgroup Conjecture [21] and an extension of the techniques of [1].

Combining the recent work of Wise and Agol with Theorem 1.2 we deduce that the fundamental group of every closed, orientable 3-manifold is conjugacy separable. As described above, conjugacy separability for any compact, orientable 3-manifold, possibly with boundary, reduces to the closed case. We therefore have a complete resolution of the question of conjugacy separability for the fundamental groups of compact, orientable 3-manifolds.

Theorem 1.3. *If M is any compact, orientable 3-manifold then $\pi_1(M)$ is conjugacy separable.*

We emphasize again that Theorem 1.3 depends on the results of [2] and [34].

1.2 Double-coset separability

In [32], the second and third authors proved a combination theorem for conjugacy-separable groups, and were able to check the hypotheses in the case when the vertex groups are the fundamental groups of Seifert-fibered 3-manifolds. To prove Theorem 1.2, we need to check the hypotheses of the combination theorem for the fundamental groups of hyperbolic 3-manifolds of finite volume. In particular, we need to prove that double cosets of peripheral subgroups of Kleinian groups of finite covolume are separable. In fact, we prove the following more general result.

Theorem 1.4 (Theorem 3.2)**.** *Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-orbifold of finite volume. If H and K are abelian subgroups of Γ and $g \in \Gamma$, then the double coset $HgK = \{hgk \mid h \in H, k \in K\}$ is separable in Γ.*

In the closed case, this result can also be deduced from the subgroup separability of Γ [2]. Indeed, Minasyan proved that, if G is a word-hyperbolic group

[1]By a *special* group we shall mean a group that is the fundamental group of a compact A-special cube complex. The reader is referred to [13] for definitions.

(as Γ is when M is closed), and every quasi-convex subgroup H is separable in G, then for any pair of quasi-convex subgroups H and K and any $g \in G$ the double coset HgK is separable [25]. (Abelian subgroups of a word-hyperbolic group are always quasi-convex.) However, Minasyan's theorem has not been generalized to the case in which M has boundary.

Haglund and Wise prove that double cosets of hyperplane subgroups are separable in virtually special groups—see [13], to which the reader is also referred for the definition of a *hyperplane subgroup*. When Γ is virtually special and H is a hyperplane subgroup of Γ then Γ has a finite-index subgroup that splits as an amalgamated product or HNN extension over H. It follows that if Γ is a Kleinian group of finite covolume then no hyperplane subgroup of Γ is abelian.

1.3 An outline

This paper is structured as follows. In Section 2 we state the algebraic results underlying the proof of Theorem 1.4. In Section 3 we prove Theorem 1.4 and discuss some implications. In Section 4 we recall the results of [32] and prove Theorem 1.2.

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2 Algebraic preliminaries

In this section we prove algebraic results that will be used in the proof of Theorem 1.4. We assume standard terminology of algebraic number theory. For reference see [19].

Notation 2.1. *By a number field we mean a finite field extension of* $\mathbb{Q}$. *If* k *is a number field, let* $\mathcal{O}_k$ *denote the ring of algebraic integers of* k. *If* $\mathfrak{p}$ *is a non-zero prime ideal of* $\mathcal{O}_k$, *then we complete* k *at* $\mathfrak{p}$ *to obtain the local field* $k_{\mathfrak{p}}$, *with ring of algebraic integers* $\mathcal{O}_{k_{\mathfrak{p}}}$. *The ring* $\mathcal{O}_{k_{\mathfrak{p}}}$ *has a unique maximal ideal. The quotient of* $\mathcal{O}_{k_{\mathfrak{p}}}$ *by this maximal ideal is called the residue class field of* $\mathcal{O}_{k_{\mathfrak{p}}}$. *The quotient map is called the residue class field map with respect to* $\mathfrak{p}$.

We begin by stating two theorems and two corollaries from [16].

Theorem 2.2. *Let* k *be a number field and let* δ *be a non-zero element of* k *that is not a root of unity. Let* S *be a finite set of prime ideals of* $\mathcal{O}_k$. *Then there exists a positive integer* n *with the following property. For each integer* $m \geq n$, *there exists a non-zero prime ideal* $\mathfrak{p}$ *of* $\mathcal{O}_k$, *lying outside of* S, *such that* $\delta \in \mathcal{O}_{k_{\mathfrak{p}}}$ *and the multiplicative order of the image of* δ *in the residue class field of* $\mathcal{O}_{k_{\mathfrak{p}}}$ *is equal to* m.

For the proof of Theorem 2.2 please see Theorem 2.3 of [16].

Corollary 2.3. *Let R be a finitely generated ring in a number field k, let δ be a non-zero element of R that is not a root of unity, and let $x_1, x_2, \ \dots \ , x_j$ be non-zero elements of R. Then there exists a positive integer n with the following property. For each integer $m \geq n$, there exist a finite field F and a ring homomorphism $\eta : R \to F$ such that the multiplicative order of $\eta(\delta)$ is equal to m and $\eta(x_i) \neq 0$, for each $1 \leq i \leq j$.*

The deduction of Corollary 2.3 from Theorem 2.2 is given in [16]. (See Corollary 2.5 of [16].) We include the proof in this paper for the convenience of the reader.

Proof. Fix a finite generating set G of R. Let S denote the finite set of prime ideals of $\mathcal{O}_{\mathrm{k}}$ which divide an element of $\{G, x_1, x_2, \ ... \ , x_j\}$. By Theorem 2.2, there exists a positive integer n with the following property. For each integer $m \geq n$ there exists a non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}_{\mathrm{k}}$, lying outside of S, such that $\delta \in \mathcal{O}_{\mathrm{k}_\mathfrak{p}}$ and the multiplicative order of the image of δ in the residue class field of $\mathcal{O}_{\mathrm{k}_\mathfrak{p}}$ is equal to m. Fix $m \geq n$ and let $\mathfrak{p} \subset \mathcal{O}_{\mathrm{k}}$ be the corresponding prime ideal. Let F denote the residue class field of $\mathcal{O}_{\mathrm{k}_\mathfrak{p}}$ and let $\eta : \mathcal{O}_{\mathrm{k}_\mathfrak{p}} \to F$ denote the residue class field map with respect to $\mathfrak{p}$. Since $\mathfrak{p} \notin S$, $R \in \mathcal{O}_{\mathrm{k}_\mathfrak{p}}$ and $x_1, x_2, \ ..., x_j$ are units in $\mathcal{O}_{\mathrm{k}_\mathfrak{p}}$. Therefore, the restriction of η to R satisfies the conclusion of the corollary. □

Theorem 2.4. *Let k be a number field. Let λ and ω be non-zero elements of k such that λ is not a multiplicative power of ω. Let P be a finite set of prime ideals of $\mathcal{O}_{\mathrm{k}}$. Then there exist primes $\mathfrak{p}$ and $\mathfrak{q}$, lying outside of P, such that $\lambda, \omega \in \mathcal{O}_{\mathrm{k}_\mathfrak{p}} \cap \mathcal{O}_{\mathrm{k}_\mathfrak{q}}$ and $(\eta_\mathfrak{p} \times \eta_\mathfrak{q})(\lambda)$ is not a multiplicative power of $(\eta_\mathfrak{p} \times \eta_\mathfrak{q})(\omega)$.*

For the proof of Theorem 2.4 please see Theorem 2.7 of [16].

Corollary 2.5. *Let R be a finitely generated ring in a number field k. Let λ and ω be non-zero elements of R such that λ is not a multiplicative power of ω. Then there exist a finite ring S and a ring homomorphism $\eta : R \to S$ such that $\eta(\lambda)$ is not a multiplicative power of $\eta(\omega)$.*

The deduction of Corollary 2.5 from Theorem 2.4 is similar to the deduction of Corollary 2.3 from Theorem 2.2. See Corollary 2.8 of [16] for details.

Theorem 2.4 can be interpreted as a multiplicative subgroup separability result for finitely generated rings lying in number fields. We conclude this section with an additive subgroup separability result.

Theorem 2.6. *Let R be a finitely generated ring in a number field k. By fixing a $\mathbb{Q}$ embedding of k into $\mathbb{C}$, we may view $k \subset \mathbb{C}$. Let β be an element of R and set $A = \{m + n\beta \mid m, n \in \mathbb{Z}\}$ and $B = \{m + n\beta \mid m, n \in \mathbb{Q}\}$. If $b \in R - B$, then there exist a finite ring S and a ring homomorphism $\eta : R \to S$ such that $\eta(b) \notin \eta(A)$.*

Proof. We first consider the case where $b \notin \mathbb{Q}(\beta)$. Let L denote the normal closure of k over $\mathbb{Q}$. Since $b \notin \mathbb{Q}(\beta)$, there exists an element $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ such that $\sigma(b) \neq b$ and σ fixes $\mathbb{Q}(\beta)$ pointwise. By the Tchebotarev Density Theorem, there are infinitely many primes p of $\mathbb{Q}$ with unramified extension $\mathfrak{p}$ in L such that σ is the Frobenius automorphism for $\mathfrak{p}/p$. Fix one such $\mathfrak{p}/p$ such that $R \subset \mathcal{O}_{\mathrm{L}_\mathfrak{p}}$, where $\mathcal{O}_{\mathrm{L}_\mathfrak{p}}$ denotes the ring of integers in the $\mathfrak{p}$-adic field $L_\mathfrak{p}$. Such a choice is possible since R is finitely generated, and each generator is an integer in $L_\mathfrak{p}$ for all but finitely many $\mathfrak{p}$. Let $F_\mathfrak{p}$ denote the residue class field of $\mathcal{O}_{\mathrm{L}_\mathfrak{p}}$ and let $\mathbb{F}_p$ denote the finite field of p elements. Let η be the composition of the inclusion map of R into $\mathcal{O}_{\mathrm{L}_\mathfrak{p}}$ with the residue map:

$$\eta : R \hookrightarrow \mathcal{O}_{\mathrm{L}_\mathfrak{p}} \to F_\mathfrak{p}.$$

Since σ is the Frobenius automorphism of $L/\mathbb{Q}$ with respect to $\mathfrak{p}/p$, $\mathrm{Gal}(L_\mathfrak{p}/\mathbb{Q}_p) = \langle \sigma' \rangle$ where $\sigma' = \sigma$ on L. Since $A \subset \mathbb{Q}(\beta)$ and σ fixes $\mathbb{Q}(\beta)$ pointwise, $A \subset \mathbb{Q}_p$. Since $\sigma(b) \neq b$, $b \notin \mathbb{Q}_p$. The Galois group of $F_\mathfrak{p}/\mathbb{F}_p$ is also induced by σ. It follows that $\eta(A) \subset \mathbb{F}_p$, but $\eta(b) \notin \mathbb{F}_p$. Therefore, η and $S = \mathbb{F}_p$ satisfy the conclusion of the theorem.

Now consider the case where $b \in \mathbb{Q}(\beta)$. Let f be the minimal monic polynomial of β over $\mathbb{Q}$ and let n be the degree of f. Our assumption that $b \notin B$ implies that $n > 2$. Moreover, we can express $b = a_0 + a_1\beta + a_2\beta^2 + \ \dots\ + a_{n-1}\beta^{n-1}$, where $a_i \in \mathbb{Q}$ and at least one coefficient $a_{i_0} \in \{a_2, a_3, \ \dots\ , a_{n-1}\}$ is non-zero. Since R is a finitely generated ring consisting of algebraic numbers, R is integral over $\mathbb{Z}[1/s]$, for all but finitely many integers s. Choose s such that the coefficients of f are in $\mathbb{Z}[1/s]$ and $a_i \in \mathbb{Z}[1/s]$, $\forall i \in \{0, 1, 2, \ \dots\ , n-1\}$. Let $\mathbb{Z}[1/s][T]$ denote the polynomial ring of $\mathbb{Z}[1/s]$. Let p be a prime that does not divide s or the numerator of a_{i_0} and let $\mathbb{F}_p$ denote the finite field of p elements. Since f is a monic polynomial, the map

$$\mathbb{Z}[1/s][T] \to \mathbb{Z}[1/s][\beta], \text{ given by } T \to \beta,$$

is an epimorphism with kernel (f). Therefore, the map

$$\rho : \mathbb{Z}[1/s][\beta] \to \mathbb{Z}[1/s][T]/(f), \text{ given by } \beta \to T + (f),$$

is an isomorphism. The quotient map

$$\mathbb{Z}[1/s] \to \mathbb{Z}[1/s]/(p) \cong \mathbb{F}_p$$

induces

$$\phi : \mathbb{Z}[1/s][T]/(f) \to \mathbb{F}_p[T]/(\overline{f}),$$

where $\overline{f}$ is the image of f in the polynomial ring $\mathbb{F}_p[T]$. Write $\overline{f} = \overline{f}_1\overline{f}_2 \ \dots\ \overline{f}_m$ as a product of irreducible factors in $\mathbb{F}_p[T]$, and let $F_i = \mathbb{F}_p[T]/(\overline{f}_i)$. Then the natural map

$$\psi : \mathbb{F}_p[T]/(\overline{f}) \to F_1 \times F_1 \times \ \dots\ \times F_m$$

is an isomorphism. Let

$$\rho_i : \mathbb{Z}[1/s][\beta] \to \mathbb{Z}[1/s][T]/(f) \to \mathbb{F}_p[T]/(\overline{f}) \to F_1 \times F_1 \times \ \dots\ \times F_m \to F_i$$

be the composition of $(\psi\phi\rho)$ with the projection of $F_1 \times F_1 \times \ \dots \ \times F_m$ onto F_i. Since F_i is a field, the kernel of ρ_i is a maximal ideal $\mathfrak{p}_i$ of $\mathbb{Z}[1/s][\beta]$. By replacing R with $R[1/s]$, if necessary, we may assume that $1/s \in R$. Thus, R is an integral extension of $\mathbb{Z}[1/s][\beta]$. Therefore, there exists a maximal ideal $\mathfrak{q}_i$ of R such that $(\mathbb{Z}[1/s][\beta]) \cap \mathfrak{q}_i = \mathfrak{p}_i$. Let η_i denote the quotient map $R \to R/\mathfrak{q}_i$. Note that the restriction of η_i to $\mathbb{Z}[1/s][\beta]$ is equal to ρ_i. Let $\pi_i : R/\mathfrak{q}_1 \times R/\mathfrak{q}_2 \times \ \dots \ \times R/\mathfrak{q}_m \to R/\mathfrak{q}_i$ denote the projection onto the i-th factor. By the universal property of direct products of rings, there exists a unique ring homomorphism $\eta : R \to R/\mathfrak{q}_1 \times R/\mathfrak{q}_2 \times \ \dots \ \times R/\mathfrak{q}_m$ such that $\pi_i \circ \eta = \eta_i$. We claim that $\eta(b) \notin \eta(A)$. Assume, to the contrary, that $\eta(b) = \eta(c_0 + c_1\beta)$ for some $c_0 + c_1\beta \in A$. Then

$$b - (c_0 + c_1\beta) = (a_0 - c_0) + (a_1 - c_1)\beta + a_2\beta^2 + \ \dots \ + a_{n-1}\beta^{n-1}$$

is in the kernel of ρ_i for every $i \in \{1, 2, \ \dots \ , m\}$. Let

$$h(T) = (a_0 - c_0) + (a_1 - c_1)T + a_2T^2 + \ \dots \ + a_{n-1}T^{n-1} \in \mathbb{Z}[1/s][T]$$

and let $\overline{h}$ be the image of h in $\mathbb{F}_p[T]$. Then $\overline{f}$ divides $\overline{h}$ in $\mathbb{F}_p[T]$. But this contradicts the fact that the image of a_{i_0} is non-zero in $\mathbb{F}_p$ and so $1 < \deg(\overline{h}) < n$. Therefore, η and $S = R/\mathfrak{q}_1 \times R/\mathfrak{q}_2 \times \ \dots \ \times R/\mathfrak{q}_m$ satisfy the conclusion of the theorem. □

3 Proof of double-coset separability

In this section we prove that double cosets of abelian subgroups of Kleinian groups of finite covolume are separable. The proof will use the following proposition from [27].

Proposition 3.1. *Let G_0, H and K be finitely generated subgroups of a group G and set $H_0 = H \cap G_0$ and $K_0 = K \cap G_0$. If $[G : G_0]$ is finite and if H_0K_0 is separable in G_0, then HK is separable in G.*

For a proof of this result see Proposition 2.2 of [27].

Theorem 3.2. *Let $M = \mathbb{H}^3/\Gamma$ be an orientable hyperbolic 3-orbifold of finite volume. Let H and K be abelian subgroups of Γ, and let $g \in \Gamma$. Then the double coset $HgK = \{hgk \mid h \in H, k \in K\}$ is separable in Γ.*

Proof. As noted in [27], since the profinite topology on Γ is equivariant under left and right multiplication, the double coset HgK is closed in Γ if and only if $H^gK = g^{-1}HgK$ is closed in Γ. Therefore, to prove the theorem, it suffices to show that if H and K are abelian subgroups of Γ, then the double coset HK is separable in Γ. Note that $g \in HK$ if and only if $g^{-1} \in KH$. Therefore, HK is separable in Γ if and only if KH is separable in Γ.

By Selberg's Lemma [4], Γ has a subgroup of finite index Γ_0 which is torsion free. Let $H_0 = H \cap \Gamma_0$ and $K_0 = K \cap \Gamma_0$. By Proposition 3.1, if H_0K_0 is separable in Γ_0, then HK is separable in Γ. Therefore, we may assume that Γ is torsion free, and thus the fundamental group of a hyperbolic manifold. Since abelian subgroups of finitely generated Kleinian groups are separable [3], we may assume that H and K do not commute. In particular, both H and K are non-trivial. Since Γ is torsion free, the non-trivial abelian subgroups of Γ are free abelian of rank 1 or 2. The free abelian subgroups of rank 1 can be generated by loxodromic or parabolic isometries. The free abelian subgroups of rank 2 are generated by parabolic isometries. If H has rank 1, let H' be the maximal cyclic subgroup of Γ containing H. If H has rank 2, let H' be the maximal abelian subgroup of Γ containing H. Then H has finite index in H'. Let $\{a_1, a_2, \ldots, a_{n-1}\}$ be a set of non-trivial coset representatives of H'/H. By [3], H is a separable subgroup of Γ. Therefore, there exists a subgroup Γ_H of finite index in Γ such that $H \subset \Gamma_H$ but $\Gamma_H \cap \{a_1, a_2, \ldots, a_{n-1}\} = \emptyset$. Then $\Gamma_H \cap H' = H$. In a similar way, define K' and choose Γ_K of finite index in Γ such that $\Gamma_K \cap K' = K$. Let $\Gamma_0 = \Gamma_H \cap \Gamma_K$ and set $H_0 = H \cap \Gamma_0$ and $K_0 = K \cap \Gamma_0$. Then Γ_0 is a subgroup of finite index in Γ, $H_0 = \Gamma_0 \cap H'$ and $K_0 = \Gamma_0 \cap K'$. By Proposition 3.1, by replacing Γ with Γ_0, if necessary, we may assume that rank-one elements of $\{H, K\}$ are maximal cyclic subgroups of Γ and rank-two elements of $\{H, K\}$ are maximal abelian subgroups of Γ. This assumption will be used if H and/or K is parabolic. However, the proof does not require loxodromic subgroups to be maximal.

Given the assumptions and reductions above, we need to consider the following cases.

Case 1. H loxodromic, K loxodromic

The group of orientation preserving isometries of $\mathbb{H}^3$ may be identified with $\mathrm{PSL}(2,\mathbb{C})$. Thus there exists a discrete, faithful representation $\rho : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ which is well-defined up to conjugation in $\mathrm{PSL}(2,\mathbb{C})$. Let $\mathbb{Q}(\mathrm{tr}\Gamma)$ denote the field obtained by adjoining the traces of the elements of $\rho(\Gamma)$ to $\mathbb{Q}$. Since M has finite volume, it follows from Mostow Rigidity that $\mathbb{Q}(\mathrm{tr}\Gamma)$ is a number field. By Proposition 2.2(e) of [5] we may conjugate $\rho(\Gamma)$ in $\mathrm{PSL}(2,\mathbb{C})$ to lie in a finite field extension of $\mathbb{Q}(\mathrm{tr}\Gamma)$. Therefore we may view $\Gamma \subset \mathrm{PSL}(2,k)$, where k is a finite extension of $\mathbb{Q}$. Write $H = \langle h \rangle$ and $K = \langle k \rangle$. In this case, h and k are conjugate in $\mathrm{PSL}(2,\mathbb{C})$ to matrices of the form

$$\pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},\ \lambda \in \mathbb{C},\ |\lambda| \neq 1,\ \text{and}\ \pm \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix},\ \omega \in \mathbb{C},\ |\omega| \neq 1,$$

respectively. Since $\mathbb{Q}(\mathrm{tr}\Gamma)$ is a number field, the eigenvalues, λ and ω, are algebraic numbers. By adjoining λ and ω to k, if necessary, we may assume that h and k are diagonalizable over k. Therefore, after conjugating in $\mathrm{PSL}(2,k)$,

$$h = \pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ and } k = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1},$$

for some

$$g = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,k).$$

By assumption, h and k do not commute. Since Γ is discrete, it follows that the fixed points of h on the sphere at infinity are disjoint from the fixed points of k. Therefore, all of the elements in $\{a, b, c, d\}$ are non-zero.

Let G be the subgroup of $\mathrm{PSL}(2,k)$ generated by Γ and g. We will show that HKg is separable in G. Since the profinite topology on G is equivariant under left and right multiplication, it will follow that HK is separable in G, implying that HK is separable in Γ, as required. To prove that HKg is separable in G, note that

$$\begin{aligned} HKg = \Big\{ \pm \begin{pmatrix} \lambda^m & 0 \\ 0 & \lambda^{-m} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega^n & 0 \\ 0 & \omega^{-n} \end{pmatrix} \\ = \pm \begin{pmatrix} a\lambda^m\omega^n & b\lambda^m\omega^{-n} \\ c\lambda^{-m}\omega^n & d\lambda^{-m}\omega^{-n} \end{pmatrix} \ \Big| \ m, n \in \mathbb{Z} \Big\}, \end{aligned}$$

and let

$$\gamma = \pm \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in G - HKg$$

be given. Since M has finite volume, Γ is finitely generated. Therefore, G is finitely generated. Let R be the ring generated by the coefficients of G over $\mathbb{Z}$. Then $G \subset \mathrm{PSL}(2,R) \subset \mathrm{PSL}(2,k)$. Suppose that $\mathrm{rs/ab}$ is not a multiplicative power of λ^2. Then by Corollary 2.5, there exist a finite ring S and a ring homomorphism $\eta : R \to S$ such that $\eta(ab) \neq 0$ and $\eta(\mathrm{rs/ab})$ is not a multiplicative power of $\eta(\lambda^2)$. The map η induces a group homomorphism

$$\overline{\eta} : G \hookrightarrow \mathrm{PSL}(2,R) \to \mathrm{PSL}(2,S).$$

Suppose that $\overline{\eta}(\gamma) \in \overline{\eta}(HKg)$. Then there exist elements $m, n \in \mathbb{Z}$ such that $\overline{\eta}(\gamma) = \overline{\eta}(h^m k^n g)$. Equating coefficients, we have:

$$\left\{ \begin{array}{ll} \eta(r) = \eta(a\lambda^m\omega^n), & \eta(s) = \eta(b\lambda^m\omega^{-n}), \\ \eta(t) = \eta(c\lambda^{-m}\omega^n), & \eta(u) = \eta(d\lambda^{-m}\omega^{-n}) \end{array} \right\}$$

or

$$\left\{ \begin{array}{ll} \eta(-r) = \eta(a\lambda^m\omega^n), & \eta(-s) = \eta(b\lambda^m\omega^{-n}), \\ \eta(-t) = \eta(c\lambda^{-m}\omega^n), & \eta(-u) = \eta(d\lambda^{-m}\omega^{-n}) \end{array} \right\}.$$

In either case, $\eta(\mathrm{rs}) = \eta(\mathrm{ab}\lambda^{2m})$, a contradiction. Therefore, we may assume that $\mathrm{rs/ab} = \lambda^{2m_0}$, for some $m_0 \in \mathbb{Z}$. By a similar argument, we may assume that $\mathrm{rt/ac} = \omega^{2n_0}$, for some $n_0 \in \mathbb{Z}$.

By assumption, $\gamma \notin HKg$. In particular,

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} \notin \Big\{ \pm \begin{pmatrix} a\lambda^{m_0}\omega^{n_0} & b\lambda^{m_0}\omega^{-n_0} \\ c\lambda^{-m_0}\omega^{n_0} & d\lambda^{-m_0}\omega^{-n_0} \end{pmatrix} \Big\}.$$

Since R is finitely generated, $R \subset \mathcal{O}_{\mathrm{k}_\mathfrak{p}}$ for all but finitely many prime ideals $\mathfrak{p}$ of $\mathcal{O}_\mathrm{k}$. For each of these primes $\mathfrak{p}$ the residue class field map $\eta_\mathfrak{p} : \mathcal{O}_{\mathrm{k}_\mathfrak{p}} \to F_\mathfrak{p}$ induces a group homomorphism

$$\overline{\eta}_\mathfrak{p} : G \hookrightarrow \mathrm{PSL}(2, \mathcal{O}_{\mathrm{k}_\mathfrak{p}}) \to \mathrm{PSL}(2, F_\mathfrak{p}),$$

where $F_{\mathfrak{p}}$ is the residue class field of $\mathcal{O}_{k_{\mathfrak{p}}}$. Choose $\mathfrak{p}$ such that

$$(*)\quad \begin{pmatrix} \eta_{\mathfrak{p}}(r) & \eta_{\mathfrak{p}}(s) \\ \eta_{\mathfrak{p}}(t) & \eta_{\mathfrak{p}}(u) \end{pmatrix} \notin \left\{ \pm \begin{pmatrix} \eta_{\mathfrak{p}}(a\lambda^{m_0}\omega^{n_0}) & \eta_{\mathfrak{p}}(b\lambda^{m_0}\omega^{-n_0}) \\ \eta_{\mathfrak{p}}(c\lambda^{-m_0}\omega^{n_0}) & \eta_{\mathfrak{p}}(d\lambda^{-m_0}\omega^{-n_0}) \end{pmatrix} \right\}.$$

Suppose $\overline{\eta}_{\mathfrak{p}}(\gamma) \in \overline{\eta}_{\mathfrak{p}}(HKg)$. Then there exist elements $m, n \in \mathbb{Z}$ such that $\overline{\eta}_{\mathfrak{p}}(\gamma) = \overline{\eta}_{\mathfrak{p}}(h^m k^n g)$. Equating coefficients as above, $\eta_{\mathfrak{p}}(\lambda^{2m}) = \eta_{\mathfrak{p}}(\mathrm{rs/ab}) = \eta_{\mathfrak{p}}(\lambda^{2m_0})$ and $\eta_{\mathfrak{p}}(\omega^{2n}) = \eta_{\mathfrak{p}}(\mathrm{rt/ac}) = \eta_{\mathfrak{p}}(\omega^{2n_0})$. Therefore, $\eta_{\mathfrak{p}}(\lambda^m) = \pm\eta_{\mathfrak{p}}(\lambda^{m_0})$ and $\eta_{\mathfrak{p}}(\omega^n) = \pm\eta_{\mathfrak{p}}(\omega^{n_0})$, contradicting $(*)$. This completes the proof in this case.

Case 2. H loxodromic, K maximal parabolic

The proof of this case follows from the proof of Lemma 3.2 in [15].

Case 3. H maximal parabolic, K maximal parabolic

Since M is a hyperbolic 3-manifold of finite volume, we may view M as the interior of a compact manifold M' with a finite number of tori boundary components. Such a manifold can be obtained from M by truncating the cusp tori. Furthermore, $\pi_1(M') \cong \pi_1(M) = \Gamma$. There is a one-to-one correspondence between the boundary tori of M' and the conjugacy classes of maximal parabolic subgroups of Γ. Suppose that H and K correspond to the same boundary component of M'. Then there exists an element $\zeta \in \Gamma$ such that $K = \zeta H \zeta^{-1}$. Since H and K do not commute, $\zeta \notin H$. Since H is a separable subgroup of Γ [22], there exist a finite group G and a group homomorphism $f : \Gamma \to G$ such that $f(\zeta) \notin f(H)$. Let Γ_0 be kernel of f, and set $H_0 = H \cap \Gamma_0$ and $K_0 = K \cap \Gamma_0$. Since Γ_0 is normal in Γ, $K_0 = \zeta H_0 \zeta^{-1}$. If there exists an element $\upsilon \in \Gamma_0$ such that $K_0 = \upsilon H_0 \upsilon^{-1}$, then $(\upsilon^{-1}\zeta)H_0(\upsilon^{-1}\zeta)^{-1} = H_0$. Therefore, $\upsilon^{-1}\zeta$ fixes the parabolic fixed point of H. Since H is a maximal parabolic subgroup of Γ, $\upsilon^{-1}\zeta \in H$. But then $f(\zeta) \in f(H)$, a contradiction. We conclude that H_0 and K_0 are not conjugate in Γ_0. By Proposition 3.1, if H_0K_0 is separable in Γ_0, then HK is separable in Γ. Therefore, by replacing Γ with Γ_0, if necessary, we may assume that H and K correspond to different boundary components of M'. Let T_1 and T_2 denote the boundary components of M' corresponding to H and K, respectively. By Thurston's Hyperbolic Dehn Surgery Theorem [6], we may choose generators h_1, h_2 for H such that M_{h_1} and M_{h_2} are complete hyperbolic 3-manifolds of finite volume, where M_{h_1} and M_{h_2} are the manifolds obtained by Dehn surgery on M' along T_1 sending h_1 and h_2, respectively, to a meridian of the attached solid torus. Similarly, we may choose generators k_1, k_2 for K such that M_{k_1} and M_{k_2} are complete hyperbolic 3-manifolds of finite volume, where M_{k_1} and M_{k_2} are obtained by Dehn surgery on M' along T_2.

For simplicity we use the fact that the representation $\rho : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$ may be lifted to a representation

$$\Gamma \to \mathrm{SL}(2, \mathbb{C}).$$

(See Proposition 3.1.1 of [12].) Therefore, we view $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$. By an argument similar to that in Case 1, we may assume that $\Gamma \subset \mathrm{SL}(2,k)$, where k is a finite field extension of $\mathbb{Q}$,

$$h_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } h_2 = \begin{pmatrix} 1 & \beta_1 \\ 0 & 1 \end{pmatrix},$$

for a fixed element $\beta_1 \in \mathbb{C} - \mathbb{R}$. If $A_1 = \{m + n\beta_1 \mid m, n \in \mathbb{Z}\}$, then

$$H = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \;\middle|\; x \in A_1 \right\}.$$

Moreover,

$$k_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}^{-1} \text{ and } k_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} 1 & \beta_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}^{-1},$$

for fixed elements

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \text{ and } \beta_2 \in \mathbb{C} - \mathbb{R}.$$

If $A_2 = \{m + n\beta_2 \mid m, n \in \mathbb{Z}\}$, then

$$K = \left\{ \begin{pmatrix} 1 - ya_1c_1 & ya_1{}^2 \\ -yc_1{}^2 & 1 + ya_1c_1 \end{pmatrix} \;\middle|\; y \in A_2 \right\}.$$

Since H and K do not commute, $c_1 \neq 0$. Note that $a_1{}^2$ and $c_1{}^2$ are elements of the coefficient field k. Since k is a number field, a_1 and c_1 are algebraic numbers. Therefore, after adjoining a_1 and c_1 to k, and conjugating Γ in $\mathrm{SL}(2,k)$ by

$$\begin{pmatrix} 1 & a_1/c_1 \\ 0 & 1 \end{pmatrix},$$

we may assume that

$$H = \left\{ \begin{pmatrix} 1 & -a_1/c_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_1/c_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \;\middle|\; x \in A_1 \right\}, \text{ and}$$

$$K = \left\{ \begin{pmatrix} 1 & -a_1/c_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - ya_1c_1 & ya_1{}^2 \\ -yc_1{}^2 & 1 + ya_1c_1 \end{pmatrix} \begin{pmatrix} 1 & a_1/c_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -yc_1{}^2 & 1 \end{pmatrix} \;\middle|\; y \in A_2 \right\}.$$

Then

$$HK = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -yc_1{}^2 & 1 \end{pmatrix} = \begin{pmatrix} 1 - xyc_1{}^2 & x \\ -yc_1{}^2 & 1 \end{pmatrix} \;\middle|\; x \in A_1,\ y \in A_2 \right\}.$$

To show that HK is separable in Γ, let

$$\gamma = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \Gamma - HK$$

be given. Suppose that $u \neq 1$. As in Case 1, there exists a finitely generated ring R_1 such that $\Gamma \subset \mathrm{SL}(2, R_1) \subset \mathrm{SL}(2, k)$. Since R_1 is finitely generated, $R_1 \subset \mathcal{O}_{k_\mathfrak{p}}$ for all but finitely many primes $\mathfrak{p}$ of $\mathcal{O}_{k_\mathfrak{p}}$. For each of these primes $\mathfrak{p}$ the residue class field map $\eta_\mathfrak{p} : \mathcal{O}_{k_\mathfrak{p}} \to F_\mathfrak{p}$ induces a group homomorphism

$$\overline{\eta}_\mathfrak{p} : \Gamma \hookrightarrow \mathrm{SL}(2, \mathcal{O}_{k_\mathfrak{p}}) \to \mathrm{SL}(2, F_\mathfrak{p}),$$

where $F_\mathfrak{p}$ is the residue class field of $\mathcal{O}_{k_\mathfrak{p}}$. Choose $\mathfrak{p}$ such that $\eta_\mathfrak{p}(u) \neq \eta_\mathfrak{p}(1)$. Then $\overline{\eta}_\mathfrak{p}(\gamma) \notin \overline{\eta}_\mathfrak{p}(HK)$, as required. Therefore, we may assume that $u = 1$. Since the determinant of γ is equal to 1,

$$\gamma = \begin{pmatrix} 1 + st & s \\ t & 1 \end{pmatrix}.$$

Let $B_1 = \{m + n\beta_1 \mid m, n \in \mathbb{Q}\}$ and $B_2 = \{m + n\beta_2 \mid m, n \in \mathbb{Q}\}$. Suppose that either $s \notin B_1$ or $-t/{c_1}^2 \notin B_2$. By Theorem 2.6, there exist a finite ring S and a ring homomorphism $\eta : R_1 \to S$ such that $\eta(s) \notin \eta(B_1)$ or $\eta(-t/{c_1}^2) \notin \eta(B_2)$, respectively. This ring homomorphism induces a group homomorphism

$$\overline{\eta} : \Gamma \hookrightarrow \mathrm{SL}(2, R_1) \to \mathrm{SL}(2, S),$$

such that $\overline{\eta}(\gamma) \notin \overline{\eta}(HK)$. Therefore, we may assume that $s \in B_1$ and $-t/{c_1}^2 \in B_2$. Moreover, since $\gamma \notin HK$, either $s \notin A_1$ or $-t/{c_1}^2 \notin A_2$. We will assume that $s \notin A_1$. (The argument if $-t/{c_1}^2 \notin A_2$ is similar, with the roles of H and K interchanged). Since $s \in B_1$, there exists a non-zero integer v_0 such that $v_0 s \in A_1$. Write $v_0 s = m_0 + n_0\beta_1$, where $m_0, n_0 \in \mathbb{Z}$. Since $s \notin A_1$, either v_0 does not divide m_0 or v_0 does not divide n_0. We will assume that v_0 does not divide m_0. (The argument if v_0 does not divide n_0 is similar, with the roles of h_1 and h_2 interchanged). Let M_{h_2} be the hyperbolic 3-manifold obtained from M' by Dehn surgery along T_1, as defined above. Let

$$\phi : \Gamma \cong \pi_1(M') \to \pi_1(M_{h_2})$$

be the homomorphism induced by inclusion. Then $\phi(H)$ is an infinite cyclic loxodromic subgroup of $\pi_1(M_{h_2})$ generated by $\phi(h_1)$, and $\phi(h_2)$ is trivial. By assumption, H and K correspond to different boundary components of M'. Therefore, $\phi(K)$ is a maximal parabolic subgroup of $\pi_1(M_{h_2})$. If $\phi(\gamma) \notin \phi(HK)$, then we are done by Case 2. Therefore, we may assume that $\phi(\gamma) = \phi(h_1^{m_1})\phi(k_\gamma)$, for some $m_1 \in \mathbb{Z}$ and $k_\gamma \in K$. Since v_0 does not divide m_0, $v_0 m_1 - m_0 \neq 0$.

Recall that $\Gamma \subset \mathrm{SL}(2, R_1) \subset \mathrm{SL}(2, k)$. Let L denote the normal closure of k over $\mathbb{Q}$ and let $\tau \in \mathrm{Gal}(L/\mathbb{Q})$ represent complex conjugation. Since $\beta_1 \in \mathbb{C} - \mathbb{R}$, $\tau(\beta_1) \neq \beta_1$. By the Tchebotarev Density Theorem, there exist infinitely many primes p of $\mathbb{Q}$ with unramified extension $\mathfrak{p}$ in L such that τ is the Frobenius automorphism for $\mathfrak{p}/p$. Fix one such $\mathfrak{p}/p$ such that p is an odd prime, p does not divide $v_0 m_1 - m_0$ and $R_1 \subset \mathcal{O}_{L_\mathfrak{p}}$, where $\mathcal{O}_{L_\mathfrak{p}}$ denotes the ring of integers in the $\mathfrak{p}$-adic field $L_\mathfrak{p}$. Let $F_\mathfrak{p}$ denote the residue class field of $\mathcal{O}_{L_\mathfrak{p}}$ and let $\mathbb{F}_p$ denote the finite field of p elements. Let η_1 be the composition of the inclusion map of R_1 into $\mathcal{O}_{L_\mathfrak{p}}$ with the residue map:

$$\eta_1 : R_1 \hookrightarrow \mathcal{O}_{L_\mathfrak{p}} \to F_\mathfrak{p}.$$

Since τ is the Frobenius automorphism of $L/\mathbb{Q}$ with respect to $\mathfrak{p}/p$, $\mathrm{Gal}(L_{\mathfrak{p}}/\mathbb{Q}_p) = \langle \tau' \rangle$ where $\tau' = \tau$ on L. Since $\tau(\beta_1) \neq \beta_1$, $\beta_1 \notin \mathbb{Q}_p$. The Galois group of $F_{\mathfrak{p}}/\mathbb{F}_p$ is also induced by τ. It follows that $\eta_1(\beta_1) \notin \mathbb{F}_p$. The map $\eta_1 : R_1 \to F_{\mathfrak{p}}$ induces a group homomorphism

$$\psi_1 : \Gamma \hookrightarrow \mathrm{SL}(2, R_1) \to \mathrm{SL}(2, F_{\mathfrak{p}}).$$

If $\psi_1(\gamma) \notin \psi_1(HK)$, then we are done. Suppose that $\psi_1(\gamma) = \psi_1(hk)$ for some $h \in H,\ k \in K$. If $h = h_1^m h_2^n$ and $x = m + n\beta_1$, then

$$\begin{pmatrix} \eta_1(1+st) & \eta_1(s) \\ \eta_1(t) & 1 \end{pmatrix} = \psi_1(\gamma) = \psi_1(hk) = \begin{pmatrix} \eta_1(1 - xy{c_1}^2) & \eta_1(x) \\ \eta_1(-y{c_1}^2) & 1 \end{pmatrix},$$

for some $y \in A_2$ corresponding to k. Therefore, $\eta_1(v_0 m + v_0 n \beta_1) = \eta_1(v_0 x) = \eta_1(v_0 s) = \eta_1(m_0 + n_0 \beta_1)$. Since $\eta_1(\beta_1) \notin \mathbb{F}_p$, the set $\{1, \eta_1(\beta_1)\}$ is linearly independent over $\mathbb{F}_p$. It follows that

$$(*)\quad v_0 m \equiv m_0 \pmod{p}.$$

Now consider the hyperbolic 3-manifold M_{h_2}. Recall that $\phi(H)$ is a loxodromic subgroup of $\pi_1(M_{h_2})$ generated by $\phi(h_1)$, $\phi(K)$ is a maximal parabolic subgroup of $\pi_1(M_{h_2})$, and $\phi(\gamma) = \phi(h_1^{m_1} k_\gamma)$, for some $m_1 \in \mathbb{Z}$ and $k_\gamma \in K$. As before, there exists a finitely generated ring R_2 in a number field F, such that $\pi_1(M_{h_2}) \subset \mathrm{SL}(2, R_2) \subset \mathrm{SL}(2, F)$. Moreover, after conjugating $\pi_1(M_{h_2})$ in $\mathrm{SL}(2, F)$, if necessary, we may assume that

$$\phi(h_1) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ and}$$

$$\phi(K) = \left\{ \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}^{-1} \;\middle|\; z \in A_3 = \{ m + n\beta_3 \mid m, n \in \mathbb{Z} \} \right\},$$

for some

$$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \mathrm{SL}(2, F) \text{ and } \beta_3 \in \mathbb{C} - \mathbb{R}.$$

Then

$$\phi(HK) = \left\{ \begin{pmatrix} \lambda^m (1 - a_2 c_2 z) & \lambda^m {a_2}^2 z \\ -\lambda^{-m} {c_2}^2 z & \lambda^{-m}(1 + a_2 c_2 z) \end{pmatrix} \;\middle|\; m \in \mathbb{Z},\ z \in A_3 \right\}.$$

Write

$$\phi(k_\gamma) = \begin{pmatrix} 1 & z_\gamma \\ 0 & 1 \end{pmatrix},\ z_\gamma \in A_3.$$

Since $\pi_1(M_{h_2})$ is discrete, $\phi(H)$ and $\phi(K)$ do not share a fixed point. Therefore, a_2 and c_2 are non-zero. By Corollary 2.3, there exist a finite field S and a ring homomorphism $\eta_2 : R_2 \to S$ such that the multiplicative order of $\eta_2(\lambda)$ is divisible by p, and $\eta_2(a_2)$ and $\eta_2(c_2)$ are non-zero in S. Since λ is a unit in

R_2, $\eta_2(\lambda) \neq 0$. Let o denote the multiplicative order of $\eta_2(\lambda)$. Consider the composition:

$$\psi_2 : \Gamma \to \pi_1(M_{h_2}) \hookrightarrow \mathrm{SL}(2, R_2) \to \mathrm{SL}(2, S),$$

where the first map $\phi : \Gamma \to \pi_1(M_{h_2})$ is induced by inclusion and the last map $\mathrm{SL}(2, R_2) \to \mathrm{SL}(2, S)$ is induced by η_2. If $\psi_2(\gamma) \notin \psi_2(HK)$, then we are done. Suppose that $\psi_2(\gamma) = \psi_2(hk)$, for some $h \in H,\ k \in K$. If $h = h_1^m h_2^n$, then

$$\psi_2(\gamma) = \begin{pmatrix} \eta_2(\lambda^{m_1}(1 - a_2c_2z_\gamma)) & \eta_2(\lambda^{m_1}{a_2}^2 z_\gamma) \\ \eta_2(-\lambda^{-m_1}{c_2}^2 z_\gamma) & \eta_2(\lambda^{-m_1}(1 + a_2c_2z_\gamma)) \end{pmatrix} =$$

$$\psi_2(hk) = \begin{pmatrix} \eta_2(\lambda^{m}(1 - a_2c_2z)) & \eta_2(\lambda^{m}{a_2}^2 z) \\ \eta_2(-\lambda^{-m}{c_2}^2 z) & \eta_2(\lambda^{-m}(1 + a_2c_2z)) \end{pmatrix},$$

for some $a \in A_3$ corresponding to k. This gives the equations:

$$\begin{aligned} \eta_2(\lambda^{m_1}(1 - a_2c_2z_\gamma)) &= \eta_2(\lambda^{m}(1 - a_2c_2z)) \\ \eta_2(\lambda^{m_1}{a_2}^2 z_\gamma) &= \eta_2(\lambda^{m}{a_2}^2 z) \\ \eta_2(-\lambda^{-m_1}{c_2}^2 z_\gamma) &= \eta_2(-\lambda^{-m}{c_2}^2 z) \\ \eta_2(\lambda^{-m_1}(1 + a_2c_2z_\gamma)) &= \eta_2(\lambda^{-m}(1 + a_2c_2z)). \end{aligned}$$

If $\eta_2(z_\gamma) = 0$, then $\eta_2(z) = 0$, and so $\eta_2(\lambda^m) = \eta_2(\lambda^{m_1})$. If $\eta_2(z_\gamma) \neq 0$, then by solving for $\eta_2(z/z_\gamma)$ in the second and third equations, we have that $\eta_2(\lambda^{2m}) = \eta_2(\lambda^{2m_1})$. In either case, $2m \equiv 2m_1 \pmod{o}$. Since p divides o and p is an odd prime, it follows that

$$(**) \quad m \equiv m_1 \pmod{p}.$$

Finally, consider the product

$$(\psi_1 \times \psi_2) : \Gamma \to \mathrm{SL}(2, F_{\mathfrak{p}}) \times \mathrm{SL}(2, S).$$

If $(\psi_1 \times \psi_2)(\gamma) \notin (\psi_1 \times \psi_2)(HK)$, then we are done. Suppose that $(\psi_1 \times \psi_2)(\gamma) = (\psi_1 \times \psi_2)(hk)$ for some $h \in H, k \in K$. Then $\psi_1(\gamma) = \psi_1(hk)$ and $\psi_2(\gamma) = \psi_2(hk)$. If $h = h_1^m h_2^n$, then by $(*)$ and $(**)$, $v_0 m \equiv m_0 \pmod{p}$ and $m \equiv m_1 \pmod{p}$. Therefore, $v_0 m_1 \equiv m_0 \pmod{p}$. This contradicts the fact that we chose p not to divide $v_0 m_1 - v_0$. This completes the proof of Case 3.

Case 4. H loxodromic, K parabolic

Let K' be the maximal parabolic subgroup of Γ containing K. By Case 2 and the assumptions at the beginning of the proof, we may assume that K is a maximal cyclic subgroup of Γ. Therefore, there exists a basis $\{k_1, k_2\}$ of K' such that $K = \langle k_1 \rangle$. Let $H = \langle h \rangle$. To prove that HK is separable in Γ, let $\gamma \in \Gamma - HK$ be given. By Case 2, we may assume that $\gamma \in HK' - HK$. Therefore, $\gamma = h^{a_0} k_1^{m_0} k_2^{n_0}$, for some $a_0, m_0, n_0 \in \mathbb{Z}$. Since $\gamma \notin HK$, $n_0 \neq 0$. As above, we may assume that $\Gamma \subset \mathrm{SL}(2, R) \subset \mathrm{SL}(2, k)$, where R is a finitely generated ring contained in a number field k. Moreover,

$$k_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } k_2 = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix},$$

for some $\beta \in \mathbb{C} - \mathbb{R}$. Since H is loxodromic, h is conjugate in $\mathrm{SL}(2,\mathbb{C})$ to an element

$$f = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \ \lambda \in \mathbb{C}, \ |\lambda| \neq 1.$$

The eigenvalue λ is an algebraic number. By adjoining λ to k, we may assume that h is diagonalizable over $\mathrm{SL}(2,k)$. Therefore,

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}, \text{ for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,k).$$

By expanding R, if necessary, we may assume that $\{\lambda, \lambda^{-1}, a, b, c, d\} \subset R$. Let L denote the normal closure of k over $\mathbb{Q}$ and let $\tau \in \mathrm{Gal}(L/\mathbb{Q})$ represent complex conjugation. As in Case 3, by the Tchebotarev Density Theorem, there exist infinitely many primes p of $\mathbb{Q}$ with unramified extension $\mathfrak{p}$ in L such that τ is the Frobenius automorphism for $\mathfrak{p}/p$. Fix one such $\mathfrak{p}/p$ such that p does not divide n_0 and $R \subset \mathcal{O}_{L_\mathfrak{p}}$, where $\mathcal{O}_{L_\mathfrak{p}}$ denotes the ring of integers in the $\mathfrak{p}$-adic field $L_\mathfrak{p}$. Let $F_\mathfrak{p}$ denote the residue class field of $\mathcal{O}_{L_\mathfrak{p}}$ and let $\mathbb{F}_p$ denote the finite field of p elements. Let η be the composition of the inclusion map of R into $\mathcal{O}_{L_\mathfrak{p}}$ with the residue map:

$$\eta : R \hookrightarrow \mathcal{O}_{L_\mathfrak{p}} \to F_\mathfrak{p}.$$

Since τ is the Frobenius automorphism of $L/\mathbb{Q}$ with respect to $\mathfrak{p}/p$, $\eta(\beta) \notin \mathbb{F}_p$. The map $\eta : R \to F_\mathfrak{p}$ induces a group homomorphism

$$\psi : \Gamma \hookrightarrow \mathrm{SL}(2,R) \to \mathrm{SL}(2,F_\mathfrak{p}).$$

Suppose that $\psi(\gamma) \in \psi(HK)$. Then $\psi(h^{a_0}k_1^{m_0}k_2^{n_0}) = \psi(h^a k_1^m)$, for some $a, m \in \mathbb{Z}$. Therefore, $\psi(k_1^{m_0-m}k_2^{n_0}) = \psi(h^{a-a_0})$. The trace of each element in K' is equal to 2. It follows that $\eta(\lambda^{a-a_0}) + 1/\eta(\lambda^{a-a_0}) = 2$, and so $\eta(\lambda^{a-a_0}) = 1$. This means that $\psi(f^{a-a_0})$ is trivial. Since h is conjugate to f in $\mathrm{SL}(2,R)$, $\psi(h^{a-a_0}) = \psi(k_1^{m_0-m}k_2^{n_0})$ is trivial. Therefore, $\eta(m_0 - m + n_0\beta) = 0$. Since $\eta(\beta) \notin \mathbb{F}_p$, the set $\{1, \eta(\beta)\}$ is linearly independent over $\mathbb{F}_p$. It follows that $\eta(n_0) = 0$. But this contradicts the fact that p does not divide n_0. We conclude that $\psi(\gamma) \notin \psi(HK)$, as required

Case 5. H parabolic, K parabolic

Let H' and K' be the maximal parabolic subgroups of Γ, containing H and K, respectively. By the assumptions at the beginning of the proof, either $H = H'$, or H is a maximal cyclic subgroup of H'. A similar statement is true for K. If $H = H'$ and $K = K'$ then we are done by Case 3. Therefore, we assume that either H or K is infinite cyclic. Choose bases $\{h_1, h_2\}$ for H' and $\{k_1, k_2\}$ for K' such that $H = \langle h_1 \rangle$ if H is infinite cyclic, and $K = \langle k_1 \rangle$ if K is infinite cyclic. To show that HK is separable in Γ, let $\gamma \in \Gamma - HK$ be given. By Case 3, we may assume that $\gamma \in H'K' - HK$. Therefore, $\gamma = h_1^{m_1}h_2^{n_1}k_1^{m_2}k_2^{n_2}$, for some $m_1, m_2, n_1, n_2 \in \mathbb{Z}$. Since $\gamma \notin HK$, either (i) H is infinite cyclic and $n_1 \neq 0$ or (ii) K is infinite cyclic and $n_2 \neq 0$. Without loss of generality, assume that (i) holds.

The argument is then very similar to the argument in Case 4. We may assume that $\Gamma \subset \mathrm{SL}(2,R) \subset \mathrm{SL}(2,k)$, where R is a finitely generated ring in a number field k. Moreover,

$$h_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ h_2 = \begin{pmatrix} 1 & \beta_1 \\ 0 & 1 \end{pmatrix}, \text{ and}$$

$$K = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1-acy & a^2y \\ -c^2y & 1+acy \end{pmatrix} \;\Big|\; y \in \{m + n\beta_2\} \right\},$$

$$\text{for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,R) \text{ and } \beta_1, \beta_2 \in \mathbb{C} - \mathbb{R}.$$

By assumption, H and K do not commute. Therefore $c \neq 0$. As in Case 4, by the Tchebotarev Density Theorem, there exist a prime p that does not divide n_1, a finite field $F_{\mathfrak{p}}$ of characteristic p, and a ring homomorphism

$$\eta : R \to F_{\mathfrak{p}}$$

such that the set $\{1, \eta(\beta_1)\}$ is linearly independent over $\mathbb{F}_p$, and $\eta(c) \neq 0$. The map η induces a group homomorphism

$$\psi : \Gamma \hookrightarrow \mathrm{SL}(2,R) \to \mathrm{SL}(2,F_{\mathfrak{p}}).$$

Suppose that $\psi(\gamma) \in \psi(HK)$. Then $\psi(h_1^{m_1} h_2^{n_1} k_1^{m_2} k_2^{n_2}) = \psi(h_1^{u_1} k_1^{u_2} k_2^{v_2})$, for some $u_1, u_2, v_2 \in \mathbb{Z}$. Therefore, $\psi(h_1^{m_1-u_1} h_2^{n_1}) = \psi(k_1^{u_2-m_2} k_2^{v_2-n_2})$. Since H is upper triangular and $\eta(c) \neq 0$, it follows that $\psi(k_1^{u_2-m_2} k_2^{v_2-n_2}) = \psi(h_1^{m_1-u_1} h_2^{n_1})$ is trivial. Therefore, $\eta(m_1 - u_1 + n_1\beta_1) = 0$. Since $\{1, \eta(\beta_1)\}$ is linearly independent over $\mathbb{F}_p$, $\eta(n_1) = 0$. But this contradicts the fact that p does not divide n_1. □

We conclude this section with some corollaries to the proof of Theorem 3.2. Since only Case 3 of Theorem 3.2 uses the full strength of the finite covolume assumption, it is natural to consider finitely generated Kleinian groups which are not necessarily of finite covolume.

Corollary 3.3. *Let Γ be a finitely generated, torsion-free Kleinian group. Given an abelian subgroup G of Γ, let $A(G)$ denote the maximal abelian subgroup of Γ containing G. Suppose that H and K are abelian subgroups of Γ such that $A(H)$ and $A(K)$ are not both free abelian of rank 2. Then the double coset HK is separable in Γ.*

Proof. If Γ is elementary, then Γ is virtually abelian and hence the result follows from Proposition 3.1 and [3]. If Γ is non-elementary, then Γ is isomorphic to a geometrically finite Kleinian group Γ_1 such that (*i*) every maximal parabolic subgroup of Γ_1 has rank 2, and (*ii*) the traces of the elements of Γ_1 are algebraic numbers. (See Theorem 1 of [3] or Theorem 4.2 of [30].) By replacing Γ with Γ_1, we assume that conditions (*i*) and (*ii*) hold for Γ. Our assumptions then imply that at least one of H or K must be loxodromic. The proof then follows from Cases 1, 2 and 4 of Theorem 3.2. □

In Theorem 3.2, we prove that certain double cosets of Kleinian groups of finite covolume are closed in the profinite topology on Γ. We now consider the congruence topology on Γ.

Definition 3.4. *Let k be a number field and let Γ be a finitely generated subgroup of $\mathrm{PSL}(2,k)$. Since Γ is finitely generated, $\Gamma \subset \mathrm{PSL}(2,R) \subset \mathrm{PSL}(2,k)$, where R is a ring obtained from $\mathcal{O}_k$ by inverting a finite number of elements. This ring R is Dedekind and, therefore, for any non-zero ideal I, the quotient R/I is finite. The quotient map $R \to R/I$ induces a* congruence homomorphism

$$\eta : \Gamma \hookrightarrow \mathrm{PSL}(2,R) \to \mathrm{PSL}(2,R/I).$$

We say that a subset X of Γ is closed in the congruence topology *on Γ if for every element $\gamma \in \Gamma - X$, there exists a congruence homomorphism η such that $\eta(\gamma) \notin \eta(X)$.*

Recall that a set $X \subset \Gamma$ is closed in the profinite topology on Γ if for every element $\gamma \in \Gamma - X$, there exist a finite group G and a group homomorphism $\phi : \Gamma \to G$ such that $\phi(\gamma) \notin \phi(X)$. For the profinite topology we consider all group homomorphisms from Γ into finite groups. For the congruence topology we consider only congruence homomorphisms. Therefore, the congruence topology is weaker than the profinite topology.

Corollary 3.5. *Let $M = \mathbb{H}^3/\Gamma$ be an orientable hyperbolic 3-orbifold of finite volume. Let $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ be a discrete, faithful representation such that the coefficient field of $\rho(\Gamma)$ is a finite field extension of $\mathbb{Q}$. Then we may view $\rho(\Gamma) \subset \mathrm{PSL}(2,R)$, where R is a finitely generated ring in a number field k. Let H and K be abelian subgroups of Γ, and let $g \in \Gamma$. The double coset HgK is closed in the congruence topology on Γ, with respect to ρ and R, if one of the following conditions is satisfied.*

- *The groups H and K are both loxodromic subgroups of Γ.*
- *Exactly one of $\{H,K\}$ is a loxodromic subgroup of Γ and the other is a maximal parabolic subgroup of Γ.*
- *Exactly one of $\{H,K\}$ is a loxodromic subgroup of Γ and the other is a maximal cyclic parabolic subgroup of Γ.*

Remark 3.6. *In Corollary 3.5, we do not require any of the loxodromic subgroups to be maximal.*

Proof. Since $M = \mathbb{H}^3/\Gamma$ is a Kleinian group of finite covolume, there exists a discrete, faithful representation from Γ into $\mathrm{PSL}(2,\mathbb{C})$ such that the coefficient field of the image of Γ is a finite field extension of $\mathbb{Q}$. By fixing one such representation ρ we may view $\Gamma \subset \mathrm{PSL}(2,R) \subset \mathrm{PSL}(2,k)$, where R is a finitely generated ring in a number field k. In the proof of Theorem 3.2, given an abelian subgroup H of Γ, we adjoin finitely many algebraic numbers to R, if necessary, and then conjugate Γ in $\mathrm{PSL}(2,R)$ such that H has a nice form. We need to justify that we can replace our original representation ρ with the new representation given by conjugation. (It is fine to replace our ring R with a larger ring since any ring homomorphism from the larger ring restricts to a ring homomorphism from R.) To see this let $\alpha \in \mathrm{PSL}(2,R)$ and consider $\Gamma' = \alpha\Gamma\alpha^{-1} \subset \mathrm{PSL}(2,R)$. Given a subset $X \subset \Gamma$ and an element $\gamma \in \Gamma - X$, let $X' = \alpha X\alpha^{-1} \subset \Gamma'$ and $\gamma' = \alpha\gamma\alpha^{-1} \in \Gamma' - X'$. Suppose there exist a finite ring S and a ring homomorphism $R \to S$, such that, under the induced group homomorphism

$$\eta : \mathrm{PSL}(2,R) \to \mathrm{PSL}(2,S),$$

$\eta(\gamma') \notin \eta(X')$. Then restricting η to Γ gives a congruence homomorphism

$$\Gamma \hookrightarrow \mathrm{PSL}(2,R) \to \mathrm{PSL}(2,S)$$

such that $\eta(\gamma) \notin \eta(X)$. We conclude that, for the purposes of our proof, it is legitimate to adjoin finitely many algebraic numbers to R and then replace Γ with a conjugate of Γ in $\mathrm{PSL}(2,R)$.

We first show that loxodromic subgroups, maximal parabolic subgroups, and maximal cyclic parabolic subgroups are closed in the congruence topology on Γ, with respect to ρ and R. This is well-known and the proof of some of the cases is in [3]. We include the proof here for the convenience of the reader.

Let A be a maximal abelian subgroup of Γ. As above, we view $\Gamma \subset \mathrm{PSL}(2,R) \subset \mathrm{PSL}(2,k)$, where R is a finitely generated ring in a number field k. After adjoining elements to R and k, if necessary, we may conjugate Γ in $\mathrm{PSL}(2,R)$ such that A is upper triangular. Let

$$\gamma = \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma - A$$

be given. Since Γ is discrete and A is a maximal abelian subgroup of Γ, γ and A do not share a fixed point on the sphere at infinity. Therefore, $c \neq 0$. Let I be an ideal of R that does not divide c and consider the congruence map

$$\eta : \Gamma \hookrightarrow \mathrm{PSL}(2,R) \to \mathrm{PSL}(2,R/I).$$

Then $\eta(\gamma) \notin \eta(A)$.

Let H be a loxodromic subgroup of Γ. Let A be the maximal abelian subgroup of Γ containing H, and fix $\gamma \in \Gamma - H$. By the case above, we may assume that $\gamma \in A - H$. Write $H = \langle h^m \rangle$, where h is a loxodromic element that generates A and m is a positive integer. Since $\gamma \in A - H$, $\gamma = h^a$, for some integer a that is not divisible by m. By Corollary 2.3, there exists a congruence map

$$\eta : \Gamma \hookrightarrow \mathrm{PSL}(2,R) \to \mathrm{PSL}(2,R/I)$$

such that the order of $\eta(h)$ is divisible by m. Then $\eta(\gamma) \notin \eta(H)$.

Let H be a maximal cyclic parabolic subgroup of Γ. Choose $k \in \Gamma$ such that $A = \langle h, k \rangle$ is the maximal abelian subgroup of Γ containing H, and fix $\gamma \in \Gamma - H$. By the case above, we may assume that $\gamma \in A - H$. Write $\gamma = h^a k^b$, $a, b \in \mathbb{Z}$. Since $\gamma \notin H$, $b \neq 0$. By Case 4 of Theorem 3.2, for infinitely many primes p, there exist congruence maps

$$\eta_p : \Gamma \hookrightarrow \mathrm{PSL}(2, R) \to \mathrm{PSL}(2, \mathbb{F}_p)$$

such that $\eta_p(A) \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. If p does not divide b, then $\eta_p(\gamma) \notin \eta_p(H)$.

Let H, K and g be given as in the statement of the corollary. Since the congruence topology on Γ is equivariant under left and right multiplication, it suffices to show that the double coset HK is closed in the congruence topology. By the argument above, we may assume that H and K do not commute. If H and K are both loxodromic, then the argument follows from Case 1 of Theorem 3.2. If H is loxodromic and K is a maximal parabolic subgroup of Γ, then the argument follows from the proof of Lemma 3.2 of [15]. If H is loxodromic and K is a maximal cyclic parabolic subgroup of Γ, then the argument follows from Case 4 of Theorem 3.2. $\square$

4 Proof of conjugacy separability

For a group G, we denote by $\widehat{G}$ the profinite completion of G. For a subgroup H of G, we denote by $\overline{H}$ the closure of H in $\widehat{G}$. The proof of Theorem 1.2 relies on the main technical theorem of [32], which we state here for convenience. We refer the reader to [32] for the definitions of the terms *efficient* and *profinitely acylindrical* used in it.

Theorem 4.1 ([32], Theorem 5.2)**.** *Let $\mathcal{G}$ be a finite graph of groups with conjugacy separable vertex groups. Let $G = \pi_1(\mathcal{G})$, and suppose that the profinite topology on G is efficient and that $\mathcal{G}$ is profinitely 2-acylindrical. For any vertex v of $\mathcal{G}$ and incident edges e and f, suppose furthermore that the following conditions hold:*

1. *for any $g \in G_v$ the double coset $G_e g G_f$ is separable in G_v;*
2. *the edge group G_e is conjugacy distinguished in G_v;*
3. *the intersection of the closures of G_e and G_f in the profinite completion of G_v is equal to the profinite completion of their intersection. That is, the natural map $\widehat{G_e \cap G_f} \to \overline{G}_e \cap \overline{G}_f$ is an isomorphism.*

Then G is conjugacy separable.

Definition 4.2. *A subgroup $H \subseteq G$ is called* conjugacy distinguished *if, whenever $g \in G$ is not conjugate into H, there is a finite quotient of G in which the image of g is not conjugate into the image of H.*

As in [32], we will apply Theorem 4.1 to the torus decomposition of M. Henceforth, M denotes a closed, orientable, Haken 3-manifold. Let $G = \pi_1(M)$ and let $\mathcal{G}$ be the graph-of-groups decomposition of G induced by the torus decomposition of M.

Theorem 4.3 ([32], Theorem A)**.** *For M as above, the profinite topology of the fundamental group of the graph of groups $\mathcal{G}$ is efficient.*

It is convenient to make the extra assumption that every Seifert-fibered piece of the torus decomposition of M is *large*—that is, has a fundamental group that virtually surjects a non-abelian free group. In this case, it turns out that $\mathcal{G}$ is profinitely 2-acylindrical, and so Theorem 4.1 applies. In [32], the remaining hypotheses of Theorem 4.1 were checked for graph manifolds. We will prove the corresponding results about the fundamental groups of hyperbolic manifolds.

Therefore, we need to consider $G_v = \Gamma$, the fundamental group of a finite-volume hyperbolic 3-manifold N. The incident edge groups G_e and G_v are maximal parabolic subgroups of Γ, which we shall denote P and Q. The next lemma is a consequence of Theorem 1.4.

Lemma 4.4. *For any $g \in \Gamma$, the double coset PgQ is separable in Γ.*

Lemma 4.5. *The subgroup P is conjugacy distinguished in Γ.*

Proof. As in the proof of Theorem 3.2, we may view $\Gamma \subset \mathrm{SL}(2, R) \subset \mathrm{SL}(2, k)$, where R is a finitely generated ring contained in a number field k. Since R is finitely generated, $R \subset \mathcal{O}_{\mathrm{k}_\mathfrak{p}}$ for all but finitely many primes $\mathfrak{p}$ of $\mathcal{O}_{\mathrm{k}_\mathfrak{p}}$. For each of these primes $\mathfrak{p}$ the residue class field map $\eta_\mathfrak{p} : \mathcal{O}_{\mathrm{k}_\mathfrak{p}} \to F_\mathfrak{p}$ induces a group homomorphism

$$\overline{\eta}_\mathfrak{p} : \Gamma \hookrightarrow \mathrm{SL}(2, \mathcal{O}_{\mathrm{k}_\mathfrak{p}}) \to \mathrm{SL}(2, F_\mathfrak{p}),$$

where $F_\mathfrak{p}$ is the residue class field of $\mathcal{O}_{\mathrm{k}_\mathfrak{p}}$.

Let γ be an element of Γ that is not conjugate into P. If γ is loxodromic, then the square of the trace of γ, $\mathrm{tr}(\gamma)^2$, is not equal to 4. Choose a prime $\mathfrak{p}$ of $\mathcal{O}_{\mathrm{k}_\mathfrak{p}}$ such that $R \subset \mathcal{O}_{\mathrm{k}_\mathfrak{p}}$ and $\eta_\mathfrak{p}(\mathrm{tr}(\gamma)^2) \neq 4$. Then $\overline{\eta}_\mathfrak{p}(\gamma) \notin \overline{\eta}_\mathfrak{p}(P)$, as required. Now suppose that γ is parabolic. As discussed in Case 3 of Theorem 3.2, we may view N as the interior of a compact manifold N' with a finite number of tori boundary components. There is a one to one correspondence between the boundary components of N' and the conjugacy classes of maximal parabolic subgroups of Γ. Let T_γ be the boundary component corresponding to γ and let T_P be the boundary component corresponding to P. Since γ is not conjugate into P and P is a maximal parabolic subgroup, $T_\gamma \neq T_P$. By Thurston's Hyperbolic Dehn Surgery Theorem, there exists a complete hyperbolic manifold M of finite volume obtained by Dehn surgery on N' along T_γ. We may choose M such that the image of γ is non-trivial in $\pi_1(M)$. Let

$$\phi : \Gamma \cong \pi_1(N') \to \pi_1(M)$$

be the homomorphism induced by inclusion $N' \to M$. Then $\phi(P)$ is a maximal parabolic subgroup of $\pi_1(M)$ and $\phi(\gamma)$ is loxodromic. The argument then follows as above. □

Lemma 4.6. *Let P and Q be non-conjugate maximal parabolic subgroups of Γ. There exists a sequence of group homomorphisms f_n from Γ to finite groups with the following properties.*

1. *For any finite index subgroup K of P, there exists an n with $Ker(f_n) \cap P \subset K$.*

2. *For every n, the intersection of $f_n(P)$ and $f_n(Q)$ is trivial.*

Proof. As in the proof of Lemma 4.5, we may view N as the interior of a compact manifold N' with a finite number of tori boundary components, each corresponding to a conjugacy class of a maximal parabolic subgroup of Γ. Let T_P and T_Q denote the boundary components of N' corresponding to P and Q, respectively. By assumption, $T_P \neq T_Q$. Choose a basis $\{p_1, p_2\}$ of P such that N_{p_1} and N_{p_2} are complete hyperbolic 3-manifolds of finite volume, where N_{p_1} and N_{p_2} are the manifolds obtained by Dehn surgery on N' along P sending p_1 and p_2, respectively, to a meridian of the attached solid torus. Let

$$\phi : \Gamma \cong \pi_1(N') \to \pi_1(N_{p_1})$$

be the homomorphism induced by inclusion. Then $\phi(Q)$ is a maximal parabolic subgroup of $\pi_1(N_{p_1})$, and $\phi(P)$ is a loxodromic subgroup of $\pi_1(N_{p_1})$ generated by $\phi(p_2)$. We then proceed as in Case 3 of Theorem 3.2. We may view $\pi_1(N_{p_1}) \subset \mathrm{SL}(2, R_1)$, where R_1 is a finitely generated ring in a number field. Moreover, we may assume that

$$\phi(p_2) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ and}$$

$$\phi(Q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \;\middle|\; z \in A = \{m + n\beta \mid m, n \in \mathbb{Z}\} \right\},$$

for some

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, R_1), \text{ and } \lambda,\ \beta \in R_1.$$

Since $\pi_1(N_{p_1})$ is discrete, $\phi(P)$ and $\phi(Q)$ do not share a fixed point. Therefore, a and c are non-zero. Fix a natural number n. By Corollary 2.3, there exist a finite field F_n and a ring homomorphism $\eta_n : R_1 \to F_n$ such that the multiplicative order of $\eta_n(\lambda)$ is divisible by n, $\eta_n(a) \neq 0$ and $\eta_n(c) \neq 0$. This ring homomorphism induces a group homomorphism

$$\overline{\eta}_n : \pi_1(N_{p_1}) \hookrightarrow \mathrm{SL}(2, R_1) \to \mathrm{SL}(2, F_n)$$

such that the order of $\overline{\eta}_n(\phi(p_2))$ is divisible by n. Moreover, since $\eta_n(a) \neq 0$ and $\eta_n(c) \neq 0$, the intersection of $\eta_n(\phi(\mathrm{P}))$ and $\eta_n(\phi(Q))$ is trivial. Let

$$g_n : \Gamma \cong \pi_1(N') \to \pi_1(N_{p_1}) \to \mathrm{SL}(2, F_n)$$

denote the composition $\overline{\eta}_n \circ \phi$. Then the intersection of $g_n(P)$ and $g_n(Q)$ is trivial, $g_n(p_1)$ is trivial, and $g_n(p_2)$ has order divisible by n. By a similar argument, there exist a finite field L_n and a group homomorphism

$$h_n : \Gamma \cong \pi_1(N') \to \pi_1(N_{p_2}) \to \mathrm{SL}(2, L_n)$$

such that the intersection of $h_n(P)$ and $h_n(Q)$ is trivial, $h_n(p_2)$ is trivial, and $h_n(p_1)$ has order divisible by n. Let $K_n = \mathrm{Ker}(g_n) \cap \mathrm{Ker}(h_n)$, and let

$$f_n : \Gamma \to \Gamma/K_n$$

denote the quotient map. Then the intersection of $f_n(P)$ and $f_n(Q)$ is trivial, and $\mathrm{Ker}(f_n) \cap P = K_n \cap P \subset \langle np_1, np_2 \rangle = nP$. The collection $\{f_n\}$ satisfies the conditions above, since given a subgroup K of finite index in P, there exists a natural number n, such that $nP \subset K$. □

Lemma 4.7. *Let P, Q be distinct maximal parabolic subgroups of Γ. The intersection of the closures $\overline{P} \cap \overline{Q}$ is trivial in the profinite completion $\widehat{\Gamma}$.*

Proof. We first consider the case in which P and Q are not conjugate in Γ. By the universal property of the profinite completion, the maps $f_n : \Gamma \to \Gamma/K_n$ in Lemma 4.6 extend to continuous homomorphisms $\hat{f}_n$ from $\widehat{\Gamma}$. Let $f = (\hat{f}_n) : \widehat{\Gamma} \to \prod_n \Gamma/K_n$ be the continuous homomorphism to the Cartesian product of Γ/K_n. By Lemma 4.6 its restriction to $\overline{P}$ and $\overline{Q}$ is injective and the image of $\overline{P} \cap \overline{Q}$ in Γ/K_n is trivial. Therefore $\overline{P} \cap \overline{Q}$ is trivial.

Suppose now that P and Q are conjugate in Γ. As in Case 3 of the proof of Theorem 3.2, there is a subgroup Γ_0 of finite index in Γ such that $P_0 = \Gamma_0 \cap P$ and $Q_0 = \Gamma_0 \cap Q$ are not conjugate in Γ_0. By the non-conjugate case, we have that $\overline{P}_0 \cap \overline{Q}_0 = 1$ in the profinite completion $\widehat{\Gamma}_0$. But $\widehat{\Gamma}_0$ is a subgroup of finite index in $\widehat{\Gamma}$, and so $\overline{P} \cap \overline{Q}$ is periodic in $\widehat{\Gamma}$. But $\overline{P} \cong \widehat{\mathbb{Z}^2}$, which is torsion-free, and so $\overline{P} \cap \overline{Q} = 1$ as required. □

These lemmas complement the corresponding results for Seifert-fibered manifolds, which we list below for convenience. As usual, by a *peripheral* subgroup of the fundamental group of a 3-manifold we mean a subgroup that (up to conjugacy) corresponds to a boundary component of N.

Lemma 4.8 ([27])**.** *Double cosets of peripheral subgroups in Seifert-fibered 3-manifold groups are separable.*

Lemma 4.9 ([32], Lemma 5.3)**.** *Every peripheral subgroup of the fundamental group of a Seifert-fibered 3-manifold group is conjugacy distinguished.*

Lemma 4.10 ([32], Lemma 5.4)**.** *Let N be a large Seifert-fibered 3-manifold and let P, Q be distinct peripheral subgroups of $\pi_1(N)$. Then $\widehat{P} \cap \widehat{Q} = \widehat{Z}$, the profinite completion of the canonical normal cyclic subgroup of $\pi_1(N)$.*

Proceeding exactly as in [32], Lemmas 4.7 and 4.10 can be used together to generalize Lemma 5.5 of [32] as follows.

Lemma 4.11. *Let M be a closed, orientable, Haken 3-manifold in which every Seifert-fibered piece of the torus decomposition is large. Let $G = \pi_1(M)$ and let $\mathcal{G}$ be the graph of groups induced by the torus decomposition of M. Then $\mathcal{G}$ is profinitely 2-acylindrical.*

When every Seifert-fibered piece of M is large, Theorem 1.2 is a direct consequence of Theorem 4.1, together with Theorem 4.3 and Lemmas 4.4, 4.5, 4.7, 4.8, 4.9, 4.10 and 4.11. Of course, we also need the Geometrization Theorem (proved in the Haken case by Thurston and in full by Perelman), which implies that every piece of the torus decomposition is either Seifert-fibered or hyperbolic.

Finally, proceeding exactly as in the proof of Theorem D of [32], the full statement of Theorem 1.2 follows.

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