Cohomological goodness and the profinite completion of Bianchi groups

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Abstract

The concept of cohomological goodness was introduced by J-P. Serre in his book on galois cohomology. This property relates the cohomology groups of a group to those of its profinite completion. We develop properties of goodness and establish goodness for certain important groups. We prove for example that the Bianchi groups, that is the groups $\mathrm{PSL}(2, \mathcal{O})$ where $\mathcal{O}$ is the ring of integers in an imaginary quadratic number field, are good. As an application of our improved understanding of goodness we are able to show that certain natural central extensions of Fuchsian groups are residually finite. A result which contrasts examples of P. Deligne who shows that the analogous central extensions of $\mathrm{Sp}(4, \mathbb{Z})$ do not have this property.

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1 Introduction

Let G be a group and $\widehat{G}$ its profinite completion. The group G is called *good* if the homomorphism of cohomology groups

$$H^n(\widehat{G}, M) \longrightarrow H^n(G, M)$$

induced by the natural homomorphism $G \longrightarrow \widehat{G}$ of G to its profinite completion $\widehat{G}$ is an isomorphism for every finite G-module M, see Section 3 for more detailed explanations. This important concept was introduced by J-P. Serre in [29, Section I.2.6]. In his book Serre explains the fundamental role goodness plays in the comparison of properties of a group and its profinite completion. We add here in Section 6 an interesting application of the concept. We show how goodness can be used to establish structural properties of certain naturally arising groups. In fact, the results of our Section 3 are applied in [3] together with the techniques of Section 6 to identify the fundamental groups of certain singular complex algebraic surfaces.

It is known that finitely generated free groups and surface groups are good, see Proposition 3.6. From Lemma 3.3 it follows that finitely generated virtually free groups are good and also that a succession of extensions of finitely generated free groups is good. It is however in general very difficult to say which group is good and which is not. It is for example an important open question whether the mapping class groups are good.

In our paper we prove this property for a particularly important class of arithmetic Kleinian groups: Bianchi groups that are defined as $\mathrm{PSL}(2, \mathcal{O}_d)$, where $\mathcal{O}_d$ is the ring of integers in the imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$ ($d \in \mathbb{Z}$, $d \geq 1$, square free). One of our main results is:

Theorem 1.1. *The Bianchi groups are good.*

Goodness is preserved by commensurability (see Section 3). By the classification of arithmetically defined subgroups of $PGL_2(\mathbb{C})$, an arithmetic group which is not cocompact in $PGL_2(\mathbb{C})$ is commensurable to a Bianchi group. Thus Theorem 1.1 holds for this class of groups. Also it follows from Lemma 3.3 that all the groups $\mathrm{SL}(2, \mathcal{O}_\mathrm{d})$ are good. This gives a basis to conjecture that all Kleinian groups are good. In fact, this would follow from Thurston's conjecture that every finite volume 3-manifold is virtually fibred over a circle. Indeed, if true, then the fundamental group is virtually a cyclic extension of a surface group and hence is good by Exercise 2(b) in [29, Section I.2.6] combining with the goodness of surface groups.

Theorem 1.1 is designed to begin the study of the torsion cohomology of the Bianchi groups. In fact the goodness of these groups implies

Corollary 1.2. *The virtual cohomological dimension of the profinite completion $\widehat{\mathrm{PSL}}(2, \mathcal{O}_d)$ is equal to two. The congruence kernel $C_d \leq \widehat{\mathrm{PSL}}(2, \mathcal{O}_d)$ has cohomological dimension equal to one or two. In particular, $\widehat{\mathrm{PSL}}(2, \mathcal{O}_d)$ is virtually torsion free and C_d is torsion free.*

The first statement comes from the fact that the Bianchi groups act discontinuously on 3-dimensional hyperbolic space with a finite volume quotient which is not compact (see Section 4). The proof of the second statement is contained in Section 5 where we also explain the construction of the congruence kernel. We were not able to decide whether C_d has cohomological dimension one or two.

We observe (in Section 5) that an arithmetic group having the congruence subgroup property is not good, since the profinite completion of it is not virtually torsion free and therefore its virtual cohomological dimension is infinite.

Another class of groups proved to be good in this paper are the so called limit groups, i.e. finitely generated fully residually free groups. Limit groups play a key role in the solution of the Tarski problems (see [9], [26] and related references) that asks whether the elementary theories of non-abelian free groups of different ranks are the same and whether this theory is decidable.

Theorem 1.3. *Limit groups are good.*

We also give a new way of applying the goodness of lattices in $\mathrm{PSL}(2,\mathbb{R})$. We show that an extension of a finitely generated residually finite good group with finitely generated residually finite kernel is residually finite. As a consequence it is deduced (in Section 6) that certain natural central extensions of Fuchsian groups are residually finite. A result which contrasts examples of Deligne [4] and Raghunathan [21] who show that the analogous central extensions of $\mathrm{Sp}(4,\mathbb{Z})$ and other arithmetic groups do not have this property.

The idea of the proof of Theorem 1.1 is to use the fact that Bianchi groups admit a so called hierarchy, i.e. a decomposition as a tower of free amalgamated products or HNN-extensions of finitely generated subgroups starting with the trivial subgroup (we give a general definition in Section 3.2). The existence of a hierarchy for Bianchi groups as well as the concept itself comes from geometry. One uses that a torsion free subgroup of finite index of a Bianchi group is isomorphic to the fundamental group of compact 3-manifold with boundary. The hierarchy of Bianchi groups behaves well with respect to the profinite topology: we refer to this fact as the profinite topology being efficient. This follows from subgroup separability (the property of being LERF) of Bianchi groups: a deep fact with a proof based on geometry. The hierarchy of a group with efficient profinite topology is preserved in the profinite completion which allows us to use inductively the Mayer-Vietoris sequence. Thus in the most general form our result can be formulated as follows.

Theorem 1.4. *Let G be a group admitting a hierarchy such that the profinite topology on G is efficient (with respect to the given hierarchy). Then G is good.*

Remark 1.5. A. Lubotzky used in [14] the decomposition of Bianchi groups or groups commensurable with them into an amalgamated free products or HNN-extensions to deduce that these groups contain a subgroup of finite index with a free non-abelian quotient. In his construction the corresponding amalgamated subgroups or associated subgroups are closed in the congruence topology (and therefore in the profinite topology). However these decompositions however do not carry sufficient information to deduce goodness.

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2 Cohomology of profinite groups

In this section we collect some notation and well known facts concerning the cohomology of profinite groups.

Let G be a profinite group and A a discrete G-module. We define the cohomology group $H^q(G, A)$ ($q \in \mathbb{N} \cup \{0\}$) by

$$H^q(G, A) = \varinjlim H^q(G/U, A^U),$$

where U ranges over all open normal subgroups of G and A^U is the submodule of fixed points of U.

The p-cohomological dimension of a profinite group G is the lower bound of the integers n such that for every discrete torsion G-module A, and for every $q > n$, the p-primary component of $H^q(G, A)$ is null. We shall use the standard notation $\mathrm{cd}_p(G)$ for p-cohomological dimension of the profinite group G. The cohomological dimension $\mathrm{cd}(G)$ of G is defined as the supremum $\mathrm{cd}(G) = \sup_p(\mathrm{cd}_p(G))$ where p varies over all primes p.

The next proposition gives a well-known characterization for cd_p ([29, Proposition I.11 and Proposition I.21']).

Proposition 2.1. *Let G be a profinite group, p a prime and n an integer. The following properties are equivalent:*

1. $\mathrm{cd}_p(G) \leq n$,
2. $H^q(G, A) = 0$ *holds for all $q > n$ and every discrete G-module A which is a p-primary torsion module,*
3. $H^{n+1}(G, A) = 0$ *holds when A is a simple discrete G-module annihilated by p,*
4. $H^{n+1}(H, \mathbb{F}_p) = 0$ *holds for any open subgroup H of G.*

Note that if G is pro-p then there is only one simple discrete G-module annihilated by p, namely the trivial module $\mathbb{F}_p$.

3 Goodness

This section starts off with preliminary results on the concept of goodness. In section 3.2 we show that, under suitable hypotheses, amalgamated products and HNN-extensions are good.

3.1 Preliminaries

Let G be a group and $\widehat{G}$ its profinite completion. Let M be a G-module which is finite as a set. For short we say that M is a finite G-module. Since G acts as a finite group on M we obtain a natural action of the profinite completion $\widehat{G}$ on M.

Following [29, Section I.2.6] we say that a group G is *good* if the homomorphism of cohomology groups $H^n(\widehat{G}, M) \to H^n(G, M)$ induced by the natural homomorphism $G \to \widehat{G}$ of G to its profinite completion $\widehat{G}$ is an isomorphism for all n and every finite G-module M. Already in the exercises in [29, Section I.2.6] J-P. Serre gives useful properties of the concept. In the following we recall and extend the results of J-P. Serre.

The following lemma is a useful consequence of Exercise 1 in [29, Section I.2.6]. Here the finite field $\mathbb{F}_p$ (p a prime) is always considered as a trivial module for any group.

Lemma 3.1. *A group G is good if and only if*

$$\varinjlim_{N \leq_f G} H^i(N, \mathbb{F}_p) = 0$$

for all i and for all primes p, where N ranges over all subgroups of finite index.

Proof. By Exercise 1 in [29, Section I.2.6] a group G is good if and only if

$$\varinjlim_{N \leq_f G} H^i(N, M) = 0$$

for all i and every finite module M, where N ranges over all subgroups of finite index of G. Since this limit can be started with any N we may assume that M is a trivial N-module for every N. Since cohomology commutes with direct sums in the second variable, we also can assume that M is p-primary.

Suppose now

$$\varinjlim_{N \leq_f G} H^i(N, \mathbb{F}_p) = 0$$

for all i and for all primes p. We shall use an induction on the length of the composition series of M. Consider a short exact sequence of p-primary finite discrete trivial N-modules

$$\langle 0 \rangle \to B \to M \to \mathbb{F}_p \to \langle 0 \rangle.$$

By the induction hypothesis we have $\varinjlim_{N \leq_f G} H^i(N, B) = 0$ and by assumption we also have

$$\varinjlim_{N \leq_f G} H^i(N, \mathbb{F}_p) = 0.$$

Then the long exact sequence of cohomology and its naturalness show that

$$\varinjlim_{N \leq_f G} H^i(N, M) = 0$$

as required. ☐

We call two groups *commensurable* if they contain isomorphic subgroups of finite index.

Lemma 3.2. *Let G be a good group and H a group commensurable with G. Then H is good.*

Proof. By Lemma 3.1 a group G is good if and only if

$$\varinjlim_{N \leq_f G} H^i(N, \mathbb{F}_p) = 0$$

for all i and all primes p, where N ranges over all subgroups of finite index of G. Since this limit can be started with any N of finite index, the result follows. $\square$

The following is Exercise 2 (b) in [29, Section I.2.6].

Lemma 3.3. *The group H is good if there is a short exact sequence*

$$\langle 1 \rangle \longrightarrow N \longrightarrow H \longrightarrow G \longrightarrow \langle 1 \rangle$$

such that G, N are good, N is finitely generated and the cohomology groups $H^q(N, M)$ are finite for all q ($q \in \mathbb{N}$) and all finite H-modules M.

In the following we sharpen Lemma 3.3 in case of a direct product by relaxing the hypothesis.

Proposition 3.4. *Let $G = G_1 \times G_2$ be a direct product of two good groups G_1, G_2. Then G is good.*

Proof. Let N be a subgroup of finite index of G. Put $N_j = N \cap G_j$ for $j = 1, 2$. Then N contains $N_1 \times N_2$ which in turn has finite index in G. So the family of $\{N_1 \times N_2\}$ constitutes a cofinal subfamily for $\{N\}$, when N ranges over all subgroups of finite index of G.

Therefore using the Künneth formula we get

$$\varinjlim_{N \leq_f G} H^i(N, \mathbb{F}_p) = \varinjlim_{N \leq_f G} H^i(N_1 \times N_2, \mathbb{F}_p) = \varinjlim_{N \leq_f G} \bigoplus_{p+q=i} H^p(N_1, \mathbb{F}_p) \otimes H^q(N_2, \mathbb{F}_p)$$

and using that direct limits commute with direct sum (Theorem 2.8 [24] and with tensor products (Corollary 2.20 [24]) we may conclude

$$\varinjlim_{N \leq_f G} H^i(N, \mathbb{F}_p) = \bigoplus_{p+q=i} (\varinjlim_{N \leq_f G} H^p(N_1, \mathbb{F}_p)) \otimes (\varinjlim_{N \leq_f G} H^q(N_2, \mathbb{F}_p)) = 0$$

for all i and every prime p. Hence the result follows by Lemma 3.1. $\square$

3.2 Amalgamated free products and HNN-extensions

In this subsection we give sufficient conditions for an amalgamated free product and an HNN-extension of good groups to be good. We apply these results in the next section to show that the Bianchi groups are good. We shall see below that an amalgamated free product or HNN-extension of good groups are not always good (see Section 5).

We remind the reader of two basic constructions of combinatorial group theory.

Let K_1, K_2 be groups, A a subgroup of K_1 and $f : A \to K_2$ an embedding. Then the amalgamated free product $K_1 *_A K_2$ is given by the presentation

$$K_1 *_A K_2 = \langle\, K_1,\, K_2 \mid \mathrm{rel}(K_1),\, \mathrm{rel}(K_2),\, a = f(a),\ a \in A \,\rangle.$$

By this notation we mean that $K_1 *_A K_2$ is generated by K_1, K_2 and defined by the relations $\mathrm{rel}(K_1)$, $\mathrm{rel}(K_2)$ of the groups K_1, K_2 together with the extra relations $a = f(a)$, $(a \in A)$.

Let K be a group, A a subgroup of K and $f : A \to K$ a monomorphism. Then the HNN-extension $\mathrm{HNN}(K, A, f)$ is given by the presentation

$$\mathrm{HNN}(K, A, f) = \langle\, K, t \mid \mathrm{rel}(K),\, tat^{-1} = f(a),\ a \in A \,\rangle.$$

Following [32] we say that the profinite topology on an amalgamated free product $G = K_1 *_A K_2$ is *efficient* if G is residually finite, the profinite topology on G induces the full profinite topology on K_1, K_2 and A, and if K_1, K_2 and A are closed in the profinite topology on G.

Similarly, we say that the profinite topology on an HNN-extension $\mathrm{HNN}(K, A, f)$ is *efficient* if K is residually finite, the profinite topology on G induces the full profinite topology on K, A and $f(A)$, and if K, A and $f(A)$ are closed in the profinite topology on G.

We have:

Proposition 3.5. *Let G be an amalgamated product or an HNN-extension of good groups and let the profinite topology on G be efficient. Then G is good.*

Proof. We start the proof with the case of HNN-extension.

Let $G = \mathrm{HNN}(K, A, f)$ be an HNN-extension of a good group K with an associated good subgroup A such that the profinite topology of G is efficient. First note that the efficiency implies that the profinite completion $\widehat{G}$ is a profinite HNN-extension $\mathrm{HNN}(\widehat{K}, \widehat{A}, \hat{f})$, where $\hat{f} : \widehat{A} \to \widehat{K}$ is the continuous homomorphism of the completions induced by f. Moreover, this profinite HNN-extension is proper in sense of [22], i.e. $\widehat{K}, \widehat{A}$ are embedded in $\mathrm{HNN}(\widehat{K}, \widehat{A}, \hat{f})$ (cf. [32]). Consider the Mayer-Vietoris sequence associated to G and $\widehat{G}$:

$$\begin{array}{ccccccc}
H^{n-1}(A, M) & \to & H^n(G, M) & \to & H^n(K, M) & \to & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
H^{n-1}(\widehat{A}, M) & \to & H^n(\widehat{G}, M) & \to & H^n(\widehat{K}, M) & \to & \cdots
\end{array}$$

where the vertical maps are induced by the natural embedding of the groups into their profinite completions. Since A and K are good the left vertical map and the right vertical map are isomorphisms, so the middle vertical map is an isomorphism as well. Since $H^0(G,M) = M^G = M^{\widehat{G}} = H^0(\widehat{G},M)$ the result follows in case of HNN-extensions.

Next we consider the case of an amalgamated free product. Let $G = K_1 *_A K_2$ be an amalgamated free product of good groups K_1, K_2 with an amalgamated good subgroup A such that the profinite topology of G is efficient. First note that the efficiency implies that the profinite completion $\widehat{G}$ is a profinite amalgamated free product $\widehat{K}_1 \amalg_{\widehat{A}} \widehat{K}_2$. Moreover, this profinite amalgamated free product is proper in the sense of [22], i.e. $\widehat{K}_1$, $\widehat{K}_2$ and $\widehat{A}$ are embedded in $\widehat{K}_1 \amalg_{\widehat{A}} \widehat{K}_2$ (cf. [32]). Consider the Mayer-Vietoris sequence associated to G and $\widehat{G}$:

$$\begin{array}{ccccccc}
H^{n-1}(A,M) & \to & H^n(G,M) & \to & H^n(K_1,M)\oplus H^n(K_2,M) & \to & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
H^{n-1}(\widehat{A},M) & \to & H^n(\widehat{G},M) & \to & H^n(\widehat{K}_1,M)\oplus H^n(\widehat{K}_2,M) & \to & \cdots
\end{array}$$

where the vertical maps are induced by the natural embedding of the groups to their profinite completions. Since A, K_1 and K_2 are good the left vertical map and the right vertical map are isomorphisms, so the middle vertical map is an isomorphism as well. Since $H^0(G,M) = M^G = M^{\widehat{G}} = H^0(\widehat{G},M)$ the result follows. □

We shall now discuss an immediate application of the previous proposition.

Following [16] we shall call a group G to be an $\mathcal{F}$-group if G has a presentation of the form:

$$G = \langle\, a_1, b_1, \ldots, a_n, b_n,\, c_1, \ldots c_t,\, d_1, \ldots d_s \mid c_1^{e_1} = \ldots = c_t^{e_t} = 1,$$
$$d_1^{-1} \ldots d_s^{-1} c_1^{-1} \ldots c_t^{-1} [a_1, b_1] \ldots [a_n, b_n] = 1 \,\rangle$$

where n, s, $t \geq 0$, and $e_i > 1$ for $i = 1, \ldots, t$. All lattices, i.e. discrete subgroups of finite covolume, in $\mathrm{PSL}(2,\mathbb{R})$ are $\mathcal{F}$-groups; conversely almost all $\mathcal{F}$-groups appear as lattices in $\mathrm{PSL}(2,\mathbb{R})$. A torsion free $\mathcal{F}$-group Γ is called a surface group if it has $2g$ generators a_i, b_i $(i = 1, \ldots, g)$ subject to one relation $[a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1$. Surface groups are exactly the groups which appear as fundamental groups of closed surfaces. By Proposition III.7.4 in [16] any subgroup of finite index of a $\mathcal{F}$-group is again a $\mathcal{F}$-group and so a torsion free subgroup of a $\mathcal{F}$-group of finite index is a finitely generated free group or a surface group.

Proposition 3.6. *All $\mathcal{F}$-groups are good.*

Proof. By Lemma 3.2 it suffices to proof that surface groups Γ are good. Clearly, Γ admits a decomposition into a free product with amalgamation

$$\Gamma = \langle a_1, b_1 \rangle *_C \langle a_2, b_2, \ldots a_g, b_g \rangle$$

of two free groups with a cyclic amalgamation defined by

$$[a_1, b_1]^{-1} = [a_2, b_2] \cdots [a_g, b_g].$$

Using Proposition 3.5 and the subgroup separability of $\mathcal{F}$-group proved by P. Scott ([23]) it is clear that Γ is good (see Lemma 5.2 (iii) in [6] for a more detailed proof without using Scott's result). $\square$

The following concept is useful in the next section

Definition 3.7. Let G be a group. A hierarchy for G is a finite collection $\mathcal{T}_0, \ldots, \mathcal{T}_N$ of tuples of finitely generated subgroups

$$\mathcal{T}_r = (G_1^{[r]}, \ldots, G_{n_r}^{[r]}) \qquad (r = 0, \ldots, N,\ n_r \in \mathbb{N})$$

of G such that

- $\mathcal{T}_0 = (G)$,
- the coordinates of $\mathcal{T}_N$ are all trivial groups,
- for every $r \geq 0$ and $s = 1, \ldots n_r$ there exists either $1 \leq i \leq n_{r+1}$ and a subgroup F of $G_i^{[r+1]}$ which is a $\mathcal{F}$-group such that

$$G_s^{[r]} = \mathrm{HNN}(G_i^{[r+1]}, F, t)$$

or there exist $1 \leq i \neq j \leq n_{r+1}$ and a subgroup F of both $G_i^{[r+1]}$ and $G_j^{[r+1]}$ which is a $\mathcal{F}$-group such that

$$G_s^{[r]} = G_i^{[r+1]} *_F G_j^{[r+1]}.$$

We say that the profinite topology on a group G admitting such a hierarchy is *efficient* if G is residually finite and if the profinite topology on G induces the full profinite topology on all $G_s^{[r]}$ and F and if the groups $G_s^{[r]}$ and F are closed in the profinite topology of G.

A group G is called subgroup separable (or LERF) if every finitely generated subgroup H of G is closed in the profinite topology of G, i.e. is the intersection of subgroups of finite index containing it.

Theorem 3.8. *A group which admits a hierarchy (Definition 3.7) and which is subgroup separable is good.*

Proof. We use induction on the level of the hierarchy of the decomposition of the preceding theorem. If the level is 1 the result is obvious. The inductive step follows from the following consideration. The subgroup separability implies that the profinite topology of our decomposition is efficient because a finitely generated subgroup of a subgroup separable group is subgroup separable. This allows to use Proposition 3.5. $\square$

Theorem 3.8 can be applied to prove that so called limit groups, i.e. finitely generated fully residually free groups are good. A group G is called *fully residually free* if for any finite subset X of G there is an epimorphism $G \longrightarrow F$ onto a free group F whose restriction on X is injective.

Proof of Theorem 1.3: The limit groups admit a hierarchy see [25] and [8], more precisely a hierarchy, where all amalgamated and associated subgroups in forming the free products with amalgamation and the HNN-extensions are infinite cyclic. On the other hand, Henry Wilton [33] proved recently that limit groups are subgroup separable. Therefore, the result follows from Theorem 3.9. $\square$

Analyzing the proof of Theorem 3.8 one can observe that we use only the fact that the profinite topology on a group G admitting a hierarchy is efficient. Thus we can claim the following

Theorem 3.9. *Let G be a group admitting a hierarchy such that the profinite topology on G is efficient. Then G is good.*

4 Bianchi groups

In this section we prove that all Bianchi groups are good.

Fix a natural number d and let $\mathbb{Q}(\sqrt{-d}) \subset \mathbb{C}$ be the corresponding imaginary quadratic number field. Let $\mathcal{O}_d$ be its ring of integers. The groups $\mathrm{PSL}(2, \mathcal{O}_\mathrm{d})$ are traditionally called Bianchi groups. They are discrete subgroups of $\mathrm{PSL}(2, \mathbb{C})$, hence act discontinuously on the symmetric space

$$\mathbb{H}^3 := \mathrm{PSL}(2, \mathbb{C})/\mathrm{PSU}(2, \mathbb{C})$$

of $\mathrm{PSL}(2, \mathbb{C})$. For a detailed description and the basic properties of them see [5, Section 7] and [17]. Let $\Gamma \leq \mathrm{PSL}(2, \mathcal{O}_\mathrm{d})$ be a torsion free subgroup of finite index (such subgroups exist since $\mathrm{PSL}(2, \mathcal{O}_\mathrm{d})$ is finitely generated and linear). The quotient

$$X_\Gamma := \Gamma \backslash \mathbb{H}^3$$

inherits from $\mathbb{H}^3$ the structure of a 3-manifold. It is never compact, let $X_\Gamma \subset \hat{X}_\Gamma$ be its Borel-Serre compactification, see [27] for the construction. Important for us is that $\hat{X}_\Gamma$ is a compact 3-manifold with boundary consisting of a non zero but finite number of tori. We further need that the inclusion

$$X_\Gamma \subset \hat{X}_\Gamma$$

is a homotopy equivalence, see [27]. This implies that the fundamental groups of X_Γ and $\hat{X}_\Gamma$ are isomorphic. Identifying Γ with the fundamental group of X_Γ we obtain isomorphisms

$$\Gamma \cong \pi_1(X_\Gamma) \cong \pi_1(\hat{X}_\Gamma). \tag{4.1}$$

The subgroup separability of Bianchi groups has been recently established (see [13, Theorem 3.6.1]). So Theorem 1.1 follows from Theorem 3.8 and the following

Theorem 4.1. *Every Bianchi group* $\mathrm{PSL}(2, \mathcal{O}_\mathrm{d})$ *has a subgroup of finite index which admits a hierarchy (see Definition 3.7).*

Proof. We choose any torsion free subgroup $\Gamma \leq \mathrm{PSL}(2, \mathcal{O}_\mathrm{d})$ of finite index and consider the Borel-Serre compactification $\hat{X}_\Gamma$. It is well known (see [27]) that every boundary torus T of $\hat{X}_\Gamma$ is incompressible. This is implied by the fact that the natural homomorphism $\pi_1(T) \to \pi_1(\hat{X}_\Gamma)$ is an inclusion. We conclude that $\hat{X}_\Gamma$ is a Haken-3-manifold. See [7] for explanation.

By [7, Chapter IV] there is a hierarchy for $\hat{X}_\Gamma$, i.e. a chain

$$(M_0, F_0),\ (M_1, F_1), \ldots, (M_n, \emptyset) \tag{4.2}$$

with $M_0 = \hat{X}_\Gamma$ and $F_0 = T$, where M_{i+1} is a (not necessarily connected) 3-manifold obtained by cutting M_i along an incompressible, non-boundary-parallel, 2-sided surface F_i, and where M_n is a union of 3-balls.

We infer from the Seifert-van Kampen theorem that Γ admits a hierarchy. Notice that incompressibility implies π_1-injectivity (i.e. embedding of the corresponding fundamental groups) for embeddings of 2-sided surfaces ($\neq S^2$) into the 3 manifolds M_i, see [7, Lemma III.8]. □

Examining the proof above one observes that it uses separability of subgroups of finite indices of the hierarchy blocks. So for example if one finds a hierarchy whose building blocks are geometrically finite, then one can use [1] directly in the proof.

Remark 4.2. Some classes of cocompact Kleinian groups are also known to be subgroup separable see [1, Theorem 1.5]. They can be proven to be good once they are shown to be Haken manifolds. It is proved in [12] and independently [31] that the fundamental groups of compact 3-manifolds all of whose finite index subgroups have finite abelianisations (for example those hyperbolic 3-manifold groups that violate Thurston's conjecture) are good. Moreover, as was already mentioned in the introduction, goodness would follow from Thurston's virtual fibration conjecture. This gives the basis to the conjecture that all Kleinian groups are good.

Using Lemma 3.2 we get

Corollary 4.3. *A group commensurable with a Bianchi group is good.*

From this corollary we infer that several groups given by generators and relations (such as some of the tetrahedral Coxeter groups) are good. For example

Corollary 4.4. *The tetrahedral hyperbolic Coxeter groups* $\mathbf{CT}(1) - \mathbf{CT}(17)$ *are good.*

Proof. All these groups are commensurable with Bianchi groups (see [5, Section 10.4]). □

5 Arithmetic groups with the congruence subgroup property

This subsection contains many examples of S-arithmetic groups which are not good.

We shall use the standard terminology concerning S-arithmetic groups which we introduce now. Let K be a number field and $\mathcal{O}$ its ring of integers. Let S be a finite set of places of K including the set S_∞ of archimedean places. We write

$$\mathcal{O}_S := \{\, a \in K \mid \nu(a) \geq 0 \text{ for all } \nu \notin S \,\}$$

for the subring of elements of K which are integral outside of S. We define K_ν to be the completion of the field K at its place ν. If ν is a non-archimedean place we define $\mathcal{O}_\nu$ to be the completion of $\mathcal{O}$ at ν. The maximal ideal of $\mathcal{O}_\nu$ is denoted by $\mathfrak{m}_\nu$.

Let $\mathbf{G}$ be a semisimple and simply connected K-defined linear algebraic group. This means that $\mathbf{G}$ is a subgroup of $\mathrm{GL}(n, \mathbb{C})$ for some $n \in \mathbb{N}$ and is also the zero set of a bunch of polynomials with coefficients in K. Let R be a subring of the number field K. We write $\mathbf{G}(R) := \mathbf{G} \cap \mathrm{GL}(n, R)$ for the group of R-points of $\mathbf{G}$. Let $\mathfrak{a} \leq R$ be an ideal of finite index. Clearly the kernel of the entrywise reduction map

$$\mathbf{G}(R) \to \mathbf{G}(R/\mathfrak{a})$$

is a subgroup of finite index in $\mathbf{G}(R)$ (called a principal congruence subgroup). Taking the completion $\tilde{\mathbf{G}}(R)$ (the congruence completion) with respect to the topology defined by the principal congruence subgroups we obtain an exact sequence

$$\langle 1 \rangle \to \mathrm{C}(\mathbf{G}, R) \to \widehat{\mathbf{G}}(R) \to \bar{\mathbf{G}}(R) \to \langle 1 \rangle. \tag{5.1}$$

The profinite group $\mathrm{C}(\mathbf{G}, R)$ is traditionally called the congruence kernel.

Proof of Corollary 1.2: Since the goodness is preserved by commensurability, the goodness of Γ_d implies the goodness of $\mathrm{SL}(2, \mathcal{O}_d)$ and therefore the goodness of every finite index subgroup in $\mathrm{SL}(2, \mathcal{O}_d)$. Let H be a torsion free congruence subgroup of $\mathrm{SL}(2, \mathcal{O}_d)$. Then H has cohomological dimension 2. It follows that the profinite completion $\widehat{H}$ has cohomological dimension 2 (as a profinite group). Since the congruence kernel of $\mathrm{SL}(2, \mathcal{O}_d)$ is contained in $\widehat{H}$, the congruence kernel $C \leq \widehat{\Gamma}_d$ has cohomological dimension at most 2.

Proposition 5.1. *Let $\mathbf{G}$ be a semisimple and simply connected K-defined algebraic group (K a number field). Let S be a finite set of places of K including the set S_∞ of archimedean places and let $\mathcal{O}_S$ be the corresponding ring of S-integers. Suppose that $\mathbf{G}(K_s)$ is not compact for at least one $s \in S$ and also that the congruence kernel $\mathrm{C}(\mathbf{G}, \mathcal{O}_S)$ of $\mathbf{G}(\mathcal{O}_S)$ is finite. Then $\mathbf{G}(\mathcal{O}_S)$ (and any group commensurable to it) is not good.*

Examples of groups which are in the range of our proposition are

$$\mathrm{SL}\left(2, \mathbb{Z}\left[\frac{1}{p}\right]\right) \cong \mathrm{SL}(2,\mathbb{Z}) *_{\Gamma_0(p)} \mathrm{SL}(2,\mathbb{Z}) \tag{5.2}$$

where p is a prime number and

$$\Gamma_0(p) := \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; A \in \mathrm{SL}(2,\mathbb{Z}),\ p \text{ divides c} \right\}$$

embedded in the two obvious ways into $\mathrm{SL}(2,\mathbb{Z})$ (see [28]). In fact a theorem of Mennicke [18] shows that this group has the congruence subgroup property which implies that it is not good. On the other hand the constituent groups in the amalgamated product (5.2) are good. This in turn implies that the decomposition (5.2) is not efficient. Of course the congruence subgroup property valid for $\mathrm{SL}(2,\mathbb{Z}[1/\mathrm{p}])$ shows that the profinite topology of that group induces only the congruence topology on its subgroup $\mathrm{SL}(2,\mathbb{Z})$.

Examples of not good arithmetic groups of a quite different nature (lattices in anisotropic $\mathbb{Q}$-defined linear algebraic groups) can be constructed using [11].

For the proof of Proposition 5.1 we need

Lemma 5.2. *Let* $\mathbf{G}$ *be a semisimple and simply connected* K*-defined algebraic group. Let* S *be a finite set of places of* K *including the set* S_∞ *of archimedean places and let* $\mathcal{O}_S$ *be the corresponding ring of* S*-integers. Let* $\Gamma \leq \mathbf{G}(K)$ *be a subgroup commensurable with* $\mathbf{G}(\mathcal{O}_S)$*. Then the congruence completion* $\bar{\Gamma}$ *of* Γ *is not virtually torsion free.*

Proof. By the strong approximation Theorem (see Theorem 7.12 of [19]) $\bar{\Gamma}$ is an open subgroup of the product

$$\mathbf{G}(\hat{\mathcal{O}}_S) = \prod_{\nu \notin S} \mathbf{G}(\mathcal{O}_\nu).$$

Here $\hat{\mathcal{O}}_S$ stands for the completion of the ring $\mathcal{O}_S$ with respect to the topology defined by its ideals of finite index. We conclude that $\bar{\Gamma}$ contains the product

$$\prod_{\nu \notin S \cup S_0} \mathbf{G}(\mathcal{O}_\nu) \leq \prod_{\nu \notin S} \mathbf{G}(\mathcal{O}_\nu)$$

for some finite set of places S_0.

Thus it suffices to show that for infinitely many places ν we have non trivial torsion elements in $\mathbf{G}(\mathcal{O}_\nu)$. The norm (index) of the ideal $\mathfrak{m}_\nu$ is a power of the prime p, say.

Note that the kernel of the natural homomorphism

$$\mathbf{G}(\mathcal{O}_\nu) \to \mathbf{G}(\mathcal{O}_\nu/\mathfrak{m}_\nu)$$

is a pro-p group. The field $\mathbb{F}_\nu := \mathcal{O}_\nu/\mathfrak{m}_\nu$ is finite and therefore by [15, Proposition 14] $\mathbf{G}(\mathbb{F}_\nu)$ contains the multiplicative group $\mathbb{F}_\nu^*$, in particular an element of order $p-1$. Then $\mathbf{G}(\mathcal{O}_\nu)$ contains torsion elements of order prime to p, as required. $\square$

Proof of Proposition 5.1: Let Γ be a group commensurable with $\mathbf{G}(\mathcal{O}_S)$. Since goodness is preserved by commensurability we may assume that Γ is torsion free and hence of finite cohomological dimension.

Since the congruence kernel $\mathrm{C}(\mathbf{G},\Gamma)$ of Γ is finite $\widehat{\Gamma}$ has torsion if and only if $\bar{\Gamma}$ has torsion and so has infinite cohomological dimension. The result follows. $\square$

Remark 5.3. If K is a global field of positive characteristic then the statement of Theorem 5.1 is still true. Indeed, in the same vein, we can find a subgroup of finite index Γ in the S-arithmetic group in question that contains only p-torsion. However, its profinite completion (which coincides with its congruence completion) will contain an infinite torsion group with elements having orders coprime to p, so Γ is not good.

On the other hand if the congruence kernel is infinite, the question of goodness requires a separate investigation because the S-arithmetic group might not be finitely generated (as it happens in the case of arithmetic lattices in rank one algebraic groups).

6 An application of goodness

In this section we apply the concept of goodness in order to show that certain natural central extensions of Fuchsian groups are residually finite.

Proposition 6.1. *Let G be a residually finite good group and $\varphi : H \to G$ a surjective homomorphism with finite kernel K. Then H is residually finite.*

Proof. Lemma 3.3 shows that any extension of a finite group by a good group is good. Therefore, by induction, we may assume that K is a minimal normal subgroup of H. We distinguish two cases:

Case 1: K is abelian. The action of H on K by conjugation turns K into a finite G-module. The elements of $H^2(G,K)$ correspond to classes of extensions

$$\langle 1\rangle \longrightarrow K \to E \longrightarrow G \longrightarrow \langle 1\rangle$$

of G while the elements of $H^2(\widehat{G},K)$ correspond to classes of profinite extensions

$$\langle 1\rangle \longrightarrow K \to F \longrightarrow \widehat{G} \longrightarrow \langle 1\rangle$$

of $\widehat{G}$. Let $\omega : G\times G \to K$ be a 2-cocycle representing the extension $K \to H \to G$. Since the map $H^2(\widehat{G},K) \to H^2(G,K)$ induced by the inclusion $G \to \widehat{G}$ is an isomorphism we may choose a continuous 2-cocycle $\widehat{\omega} : \widehat{G} \times \widehat{G} \to K$ which restricts to ω on G. Let $\widehat{G}_\omega \to \widehat{G}$ be the corresponding group extension. There is a homomorphism $\psi : H \to \widehat{G}_\omega$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
\langle 1\rangle & \longrightarrow & K & \longrightarrow & \widehat{G}_\omega & \longrightarrow & \widehat{G} & \longrightarrow & \langle 1\rangle \\
 & & {\scriptstyle =}\uparrow & & {\scriptstyle \psi}\uparrow & & \uparrow & & \\
\langle 1\rangle & \longrightarrow & K & \longrightarrow & H & \longrightarrow & G & \longrightarrow & \langle 1\rangle.
\end{array}$$

It follows that ψ is injective and hence H is residually finite.

Case 2: K is not abelian. Here we again consider the action of H on its normal subgroup K by conjugation. This action gives rise to a homomorphism $H \to \mathrm{Aut}(K)$ of H to the automorphism group of K. Let N be the kernel of this homomorphism. Since K is finite N is a normal subgroup of finite index in H. Since K is minimally normal in H we have $N \cap K = \langle 1 \rangle$. This implies that φ maps N injectively to a subgroup of finite index in G. We infer that N and hence H are residually finite. $\square$

Corollary 6.2. *Let G be a residually finite good group and $H \to G$ a surjective homomorphism with residually finite and finitely generated kernel K. Then H is residually finite.*

Proof. Since K is finitely generated and residually finite, it has a sequence of characteristic subgroups K_n $(n \in \mathbb{N})$ of finite index such that the intersection of the K_n is trivial. The quotient groups H/K_n are residually finite for all $n \in \mathbb{N}$ by Proposition 6.1. This easily implies that H itself is residually finite. $\square$

We shall now give an application of the preceding considerations.

Let $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ be the universal covering group of $\mathrm{PSL}(2,\mathbb{R})$. The kernel Z of the covering homomorphism

$$\pi : \widetilde{\mathrm{PSL}}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$$

is infinite cyclic. Given a subgroup $\Gamma \leq \mathrm{PSL}(2,\mathbb{R})$ we define

$$\Gamma_0 := \pi^{-1}(\Gamma), \qquad \Gamma_n := \pi^{-1}(\Gamma)/nZ \quad (n \in \mathbb{N}). \tag{6.1}$$

From Proposition 3.6 and Corollary 6.2 we get

Proposition 6.3. *Let $\Gamma \leq \mathrm{PSL}(2,\mathbb{R})$ be a lattice (Fuchsian group) then Γ_0 and all Γ_n $(n \in \mathbb{N})$ are residually finite.*

Proposition 6.3 should be contrasted with examples of P. Deligne [4]. He considers subgroups of finite index in the integral symplectic group

$$\Gamma \leq \mathrm{Sp}(4,\mathbb{Z}) \leq \mathrm{Sp}(4,\mathbb{R}).$$

He shows that their inverse images in the universal cover of $\mathrm{Sp}(4,\mathbb{R})$ are not residually finite. In his arguments the congruence subgroup property of $\mathrm{Sp}(4,\mathbb{Z})$, i.e. the triviality of the congruence kernel, plays an essential role (compare Section 5). Similar results for cocompact discrete subgroups of $\mathrm{Spin}(2,n)$ are contained in [21] combined with [20]. For more on this theme see [2], [30].

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