Finite Index Subgroups of Conjugacy Separable Groups

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To D. Segal on the occasion of his 60-th birthday

Abstract

We construct an example of conjugacy separable group possessing a not conjugacy separable subgroup of finite index. We give also a sufficient condition for a conjugacy separable group to preserve this property when passing to subgroups of finite index. We establish also conjugacy separability of finitely presented residually free groups using impressive results of Bridson and Wilton [BW-07].

1 Introduction

In 1912 Max Dehn formulated three fundamental decision problems: the word problem, the conjugacy problem and the isomorphism problem. Dehn partially solved some of these problems for finitely presented groups, so this marked the birth of a new subject, the combinatorial group theory.

In 1958 Mal'cev noticed that certain residual properties of groups are connected to two of Dehn's problem. Namely in [M-58] he showed that the word problem, the conjugacy problem and the generalized word problem (the latter asks for an algorithm to decide whether a given element is in the given finitely generated subgroup) have positive solution in finitely presented residually finite, conjugacy separable and subgroup separable groups respectively.

Recall that a group G is conjugacy separable if whenever x and y are non-conjugate elements of G, there exists some finite quotient of G in which

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the images of x and y are non-conjugate and a group G is called subgroup separable if for any finitely generated subgroup H of G and an element $g \notin H$ there exists a finite quotient of G where the image of g is not in the image of H.

Note that a subgroup of finite index of a residually finite (resp. subgroup separable) group is residually finite (resp. subgroup separable). However the conjugacy separability of a finite index subgroup of conjugacy separable group was not clear. The objective of the paper is to construct an example of conjugacy separable group possessing a non conjugacy separable finite index subgroup. Our example however is countably generated, so the question is still open for finitely generated case. We also remark that as was shown in [G-86] conjugacy separability is not preserved by finite extensions.

We say that a group G is hereditarily conjugacy separable if every finite index subgroup of G is conjugacy separable. In the second part of the article we give a sufficient condition for a conjugacy separable group to be hereditarily conjugacy separable. We use this condition to prove that a finite index subgroups of direct product of limit groups are conjugacy separable.

We combine also the impressive results of Bridson and Wilton [BW-07] with conjugacy separability of limit groups established in [CZ-07] to prove conjugacy separability for finitely presented residually free groups. Finite presentability condition is essential: it is well-known that there is a finitely generated subgroup of a direct product of two free groups for which the conjugacy problem is unsolvable (see [Mi-92] for example) and so by Malcev's observation [M-58] is not conjugacy separable.

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2 Example

Let S be a subgroup in $GL_3(\mathbb{Z})$ such that

(i) S contains the center $Z = \{\pm 1\}$ and S/Z is torsion-free;

(ii) S is not closed in the congruence topology.

Following [GS-78] make S act on $M = M_3(\mathbb{Z})$ by multiplication from the right. Then the S-orbit of the identity matrix I is exactly S, thus the orbit of I is not closed in M. Let $G = M \rtimes S$ be the respective semidirect product. Then the conjugacy class of I is exactly its S-orbit and therefore is not closed. Since I is primitive in M (i.e. is not a proper power), the element I is primitive in G as well. Indeed, $(x, y)^n = (I, 1)$ for $n > 1$,

$x \in M$, $y \in S$ implies $n = 2$ and $y = \pm 1$. Then $(x, y)^2 = (x + x^y, 1) = (x \pm x, 1) = (I, 1)$, a contradiction.

Thus there are primitive elements $x = I, y \in G$ such that x e y are conjugate in $\widehat{G}$ but are not conjugate in G. Moreover, y is not conjugate to $x^{-1} = -I$, because I and $-I$ are conjugated in G. Put $G = G_0$ and define $G_1 = HNN(G_0, < x >, < y >, t_1)$. Inductively we define $G_i = HNN(G_{i-1}, < x_{i-1} >, < y_{i-1} >, t_i)$, where x_{i-1} and y_{i-1} represents arbitrary primitive elements in G_{i-1} that are conjugates in $\widehat{G_{i-1}}$ and are not conjugates in G_{i-1}.

Lemma 2.1. *Let x be a primitive element in G. Then x is primitive in G_n for every n.*

Proof. We use induction on n. If $n = 0$ there is nothing to prove. Suppose for $n - 1$ the lemma is true, so that x is primitive in G_{n-1}. Let z be an element of G_n such that $z^k = x$ for some $k \in \mathbb{N}$. But $G_n = HNN(G_{n-1}, < x_{n-1} >, < y_{n-1} >, t_i)$ so by Theorem 2.4 in [L-S-77] $z \in G_{n-1}$. Thus by induction hypotheses $k = 1$ as required. $\square$

The groups G_i constitute an inductive system with respect to inclusions. Define the group $P := \varinjlim_i G_i$.

We shall describe now $\widehat{G}_i$. Clearly, $\widehat{G}_i = HNN(\widehat{G}_{i-1}, \overline{< x_{i-1} >}, \overline{< y_{i-1} >}, t_i)$. Let $\gamma \in \widehat{G}_{i-1}$ such that $x_{i-1}^{\gamma} = y_{i-1}$. Putting $t_{1i} = t\gamma^{-1}$ we can rewrite the presentation of $\widehat{G}_i = HNN(\widehat{G}_{i-1}, \overline{< x_{i-1} >}, t_{1i}) = \overline{\langle \widehat{G}_{i-1}, t_{1i} \mid x_{i-1}^{t_{1i}} = x_{i-1} \rangle}$. Then factorization modulo the normal closure of t_{1i} gives an epimorphism $f_i : \widehat{G}_i \longrightarrow \widehat{G}_{i-1}$ such that $f_{|\widehat{G_{i-1}}} = id$.

Lemma 2.2. *There exists an epimorphism $f : \widehat{P} \longrightarrow \widehat{G}_i$ such that $f_{|\widehat{G_i}} = id$.*

Proof. From above one has the following commutative diagram

$$\begin{tikzcd}
\widehat{P} \arrow[d, "\hat{\varphi}"] & P \arrow[l] \arrow[dl, "\varphi"] \\
\widehat{G}_i & G_j \arrow[l] \arrow[u] \arrow[d] \\
& \widehat{G}_j \arrow[ul]
\end{tikzcd} ,$$

where φ is given by the universal property of the inductive limit and $\hat{\varphi}$ by the universal property of the profinite completion. Clearly $\hat{\varphi}$ is the needed epimorphism. $\square$

Proposition 2.3. *(i) The natural homomorphism that sends* $\widehat{G}_i \longrightarrow \widehat{G}_{i+1} = \overline{\langle G_i, t \mid x^t = y\rangle}$ *is injective for every* i.

(ii) G_i *is residually finite.*

Proof. By Proposition 9.4.3 (2) in [RZ-00] we have to check that for every open normal subgroup U of G_i there exists an open normal subgroup $V \leq U$ in G_i such that $(V \cap \overline{\langle x\rangle})^t = V \cap \overline{\langle y\rangle}$. However, this equality valid already for U because x and y are conjugate in $\widehat{G}_i$. This proves (i).

To prove (ii) we use induction on i. Assume that G_i is residually finite. By Proposition 9.4.3 (3) in [RZ-00] $HNN^{abs}(\widehat{G}_i, \overline{\langle x\rangle}, \overline{\langle y\rangle}, t)$ is residually finite. On the other hand, $G_{i+1} = HNN(G_i, \langle x\rangle, \langle y\rangle, t)$ embeds in $HNN^{abs}(\widehat{G}_i, \overline{\langle x\rangle}, \overline{\langle y\rangle}, t)$ and therefore is residually finite as well.

$\square$

Lemma 2.4. *For each* i*,* G *is closed in the profinite topology of* G_i.

Proof. Let $g \in \widehat{G} \cap G_i$. We have to prove that $g \in G$. We use induction on i. Suppose $\widehat{G} \cap G_{i-1} = G$. If $g \in G_{i-1}$ then $g \in G$ by the induction hypothesis. Suppose $g \in G_i \backslash G_{i-1}$. Write g in the reduced form $g = g$ as an element of $HNN^{abs}(\widehat{G}_{i-1}, \overline{\langle x\rangle}, \overline{\langle y\rangle}, t)$ and in the reduced form $g = h_1 t^{\pm 1} h_2 t^{\pm 1} \ldots$ as an element of $G_i = HNN(G_{i-1}, \langle x\rangle, \langle y\rangle, t)$, where $h_i \in G_{i-1}$. Since $G_{i-1} \cap \overline{\langle x\rangle} = \langle x\rangle$ and h_i not in $\langle x\rangle$, one sees that $h_1 t^{\pm 1} h_2 t^{\pm 1} \ldots$ is the reduced form of g as an element of $HNN^{abs}(\widehat{G}_{i-1}, \overline{\langle x\rangle}, \overline{\langle y\rangle}, t)$. Therefore, since the length of the reduced form does not depend on the choice of the form (see [L-S-77]), the length of the latter is 1, so $g \in G_{i-1}$, a contradiction. This finishes the proof. $\square$

Corollary 2.5. G *is closed in the profinite topology of* P*, i.e.* $\widehat{G} \cap P = G$.

Proof. Let $g \in \widehat{G} \cap P$. We have to prove that $g \in G$. Since $g \in P$ there exists i such that $g \in G_i$. Then by the preceding lemma $g \in G$ as needed. $\square$

Theorem 2.6. P *is conjugacy separable.*

Proof. Let g_1 and g_2 are elements of P which are conjugate in $\widehat{P}$. Then there exists i such that $g_1, g_2 \in G_i$. By Lemma 2.2 there exists $f_i : \widehat{P} \longrightarrow \widehat{G}_i$ such that $(f_i)_{|\widehat{G}_i} = id$. Hence g_1 and g_2 are conjugate in $\widehat{G}_i$ and so by the construction of P they are conjugate in G_j for some $j > i$. Thus they are conjugate in P, as needed. $\square$

The next proposition is the correct statement of proposition 2.5 (1) in [RZ-00].

Proposition 2.7. *Let G be a group that acts on a tree S, such that the stabilizer G_e is a cyclic group for each edge e. Let H be a subgroup of G_v for some $v \in S$. Assume that either (i) each G_e is finite, or (ii) the profinite topology of G induces on each G_e its full profinite topology. Then G can be represented as a fundamental group of a graph of groups $(\mathcal{G}, X)$ such that $X = S/G$, $\mathcal{G}(x) = G_s$, where $Gs = x$, and the normalizer $N_G(H) = \pi_1(\mathcal{G}', Y)$, with the set of edges $E(Y)$ a subset of $E(X)$, and $\mathcal{G}'(Y) = N_{\mathcal{G}(y)}(H)$, for all $y \in Y$. In particular, if X is a tree, then Y is a subgraph of X.*

Proof. The proof of Proposition 2.5 in [RZ-00] is precisely the proof of this proposition. □

Corollary 2.8. *Let $H = HNN(H_0, C, t)$ be a HNN-extension of residually finite group H_0 with cyclic associated subgroup C. Assume that the profinite topology of H induces the full profinite topology on C. Then the normalizer $N_H(C)$ is either $N_{H_0}(C) \amalg_C N_{H_0^t}(C)$ or $HNN(N_{H_0}(C), C, th)$, for some $h \in H_0$.*

Proof. In the context of the preceding proposition X is just one loop and therefore Y is either an edge with two vertices or a loop. In the first case one has the desired free amalgamated product and in the second case the HNN-extension. □

Define $U = \langle G, G^{t_1}, t_1^2, t_i | 1 \neq i \in I\rangle$. It is not difficult to see that $[P : U] = 2$. Put $U_n = U \cap G_n$.

Lemma 2.9. *The elements x and y are non-conjugated in U_1.*

Proof. Since $x^{t_1^2} = y^{t_1}$, the elements x and y are conjugate in U_1 iff y is conjugated to y^{t_1} in U_1. So we prove that y and y^{t_1} are non-conjugated in U_1. Suppose on the contrary $y^g = y^{t_1}$ for some $g \in U_1$. Then $g^{-1}t_1 \in C_{G_1}(y) \leq N_{G_1}(y)$. The epimorphism $f_1 : \widehat{G}_1 \longrightarrow \widehat{G} = \widehat{SL_3(\mathbb{Z}}$ shows that G_1 induces the full profinite topology on $\langle x\rangle$ and $\langle y\rangle$. Then by Corollary 2.8 the normalizer $N_{G_1}(\langle y\rangle)$ is either $N_G(\langle y\rangle)\amalg_{\langle y\rangle} N_{G^t}(\langle y\rangle)$ or $HNN(N_G(\langle y\rangle), \langle y\rangle, ht_1)$, for some $h \in G$.

We show that the second possibility does not occur here. Indeed, if it occurs then $\langle y\rangle = \langle y\rangle^{(ht_1)^{-1}} = \langle x\rangle^{h^{-1}}$, so that $y^{\pm 1} = x^{h^{-1}}$ contradicting the choice of x and y.

Thus $N_{G_1}(\langle y\rangle) = N_G(\langle y\rangle) \amalg_{\langle y\rangle} N_{G^t}(C) \leq U$, a contradiction with $g^{-1}t_1 \in N_{G_1}(y) \setminus U_1$. This finishes the proof. □

Lemma 2.10. *The elements x and y are non-conjugated in U_n, for every n.*

Proof. We use induction on n. The case $n = 1$ is the subject of Lemma 2.9. Suppose for $n-1$ the lemma is true. Note that applying the subgroup theorem for HNN-extension (see [S-77]) (or Reidermeister- Schreier method) for the subgroup U_n of $G_n = HNN(G_{n-1}, \langle x_n \rangle, \langle y_n \rangle, t_n)$ one gets $U_n = \langle U_{n-1}, t_n, t_n^{t_1} | rel(U_{n-1}), x_n^{t_n} = y_n, (x_n^{t_1})^{t_n{}^{t_1}} = y_n^{t_1} \rangle$, if $x_n, y_n \in U_{n-1}$, and $U_n = \langle U_{n-1}, t_n, t_n^{t_1} | rel(U_{n-1}), (x_n^2)^{t_n} = y_n^2 \rangle$, if $x_n, y_n \notin U_{n-1}$. Therefore U_{n-1} can be interpreted as HNN-extension with U_{n-1} as a base group and either with two stable letters $z_1 = t_n$, $z_2 = t_n^{t_1}$ or with one stable letter t_n.

Now suppose on the contrary $x^g = y$ for some $g \in U_n$. Since the lemma is true for $n-1$, $g \notin U_{n-1}$. Then $y \in U_{n-1} \cap U_{n-1}^g$ and so x and y are conjugated to an element of either $\langle x_n \rangle$ or $\langle y_n \rangle$ (we explain this in the last paragraph of the proof for the sake of not breaking the thought). Since x and y are primitive by Lemma 2.1, x and y are conjugate to either $x_n^{\pm 1}$ or to $y_n^{\pm 1}$. But $x^{\pm 1}$ and $y^{\pm 1}$ are conjugate in G_{n-1} (in fact, in G_1), it follows that x_n is conjugate to y_n in G_{n-1}, a contraction. Thus x and y are not conjugate in U_n.

To explain why x and y are conjugated to an element of either $\langle x_n \rangle$ or $\langle y_n \rangle$ we consider the canonical action of $G_n = HNN(G_{n-1}, \langle x_n \rangle, \langle y_n \rangle, t_n)$ on the tree S associated with G_n. Then G_{n-1} is the stabilizer of a vertex v and conjugates of associated subgroups are the stabilizers of edges. It follows that x fixes the vertices v and gv and therefore the path $[v, gv]$. Similarly, y fixes the vertices v and $g^{-1}v$ and therefore the path $[v, g^{-1}v]$ Thus x and y fixe some edges incident to v and therefore are conjugate in G_{n-1} to the one of associated subgroups $\langle x_n \rangle$, $\langle y_n \rangle$. □

Theorem 2.11. *U is not conjugacy separable.*

Proof. Observe that $U = \varinjlim_n U_n$ is inductive limit of U_n. So if x and y are conjugated in U, then they are conjugated in U_n, for some n. But this contradicts Lemma 2.10. The result follows. □

3 Density of centralizers and finitely presented residually free groups

The next proposition gives a sufficient condition for a conjugacy separable group to be hereditarily conjugacy separable.

Proposition 3.1. *Let G be a conjugacy separable group and suppose for every element $g \in G$,*

$$\overline{C_G(g)} = C_{\widehat{G}}(g). \tag{1}$$

Then G is hereditarily conjugacy separable.

Proof. Let H be a finite index subgroup of G. Let h_1, h_2 be elements of H such that $h_1^\gamma = h_2$ for some γ in $\widehat{H}$. Since G is conjugacy separable, there exists $g \in G_n$ such that $h_1^g = h_2$. Then $\delta := \gamma g^{-1} \in C_{\widehat{G}}(h_1)$. It follows that $g = \delta^{-1}\gamma \in C_{\widehat{G}}(h_1)\widehat{H} \cap G$. Since H is of finite index in G the set $C_G(h_1)H$ is closed in the profinite topology, i.e. $\overline{C_G(h_1)H} \cap G = C_G(h_1)H$. By hypothesis $\overline{C_G(h_1)H} = C_{\widehat{G}}(h_1)\widehat{H}$, so $C_{\widehat{G}}(h_1)\widehat{H} \cap G = C_G(h_1)H$ and therefore $g = ch$ for some $c \in C_G(h_1), h \in H$. Hence $h_1^g = h_1^h = h_2$ as needed. □

Proposition 3.2. *Direct product preserves Condition (1).*

Proof. Let H be a finite index subgroup of $G = \prod_{i=1}^n G_i$. Write $g = (g_1, \dots, g_n)$, where $g_i \in G_i$. Then $C_G(g) = \prod_{i=1}^n C_{G_i}(g_i)$, and $C_{\widehat{G}}(g) = \prod_{i=1}^n C_{\widehat{G_i}}(g_i)$. The result follows. □

Corollary 3.3. *Let $G = \prod_{i=1}^n \Lambda_i$ is a direct product of limit groups. Then G is hereditarily conjugacy separable.*

Proof. By Lemma 3.5 in [CZ-07] every limit group satisfies Condition (1). Thus the result follows from Proposition 3.2. □

A subgroup H of G is called a virtual retract if there exists a subgroup U of finite index in G together with an epimorphism $f : U \longrightarrow H$ such that $f_H = id$.

Theorem 3.4. *Let H be a hereditarily conjugacy separable group and G is a virtual retract of H. Then G is conjugacy separable.*

Proof. Let $h_1, h_2 \in G$ be elements such that $h_1 = h_2^\gamma$ for some $\gamma \in \widehat{G}$. We show that h_1 and h_2 are conjugate in G.

Since G is a virtual retract of H, there exists a finite index subgroup U of H containing G and an epimorphism $f : U \longrightarrow G$ such that $f_{|G} = id$. Therefore, $\widehat{G} = \bar{G} \leq \widehat{U}$. By hypothesis U is conjugacy separable, so h_1 and h_2 are conjugate in U. It follows that $h_1^{f(u)} = h_2$ as needed. □

Theorem 3.5. *A finitely presented residually free group is conjugacy separable.*

Proof. Let G be a finitely presented residually free group. By Claim 7.5 in [S-01] or Corollary 19 in [BMR-99] G embeds in a direct product of finitely many limit groups $H = \prod_{i=1}^{n} L_i$. Without loss of generality we may assume that G is a subdirect product of L_is. By Theorem 8 [BW-07] the finitely presented group G contains a term of the lower central series $\gamma_m(L)$ for some natural m. By Theorem B in [BW-07] every finitely presented subgroup of L is closed in the profinite topology of L, so the profinite topology of L induces the full profinite topology on G. It follows that $\bar{G} = \widehat{G} \leq \widehat{L}$. Let g_1, g_2 be elements of G such that $g_1^{\hat{g}} = g_2$ for some $\hat{g} \in \widehat{G}$. Since L is conjugacy separable, there exists $l \in L$ such that $g_1^l = g_2$. Thus $\hat{g}l^{-1} \in C_{\widehat{L}}(g_1)$ and so $\hat{c}\hat{g} = l \in C_{\widehat{L}}(g_1)\widehat{G} \cap L$ for some $\hat{c} \in C_{\widehat{L}}(g_1)$. Since the product of any two subgroups of the finitely generated nilpotent group $L/\gamma_m(L)$ is closed (it is true even for polycyclic groups, see Exercise 13 in Chapter 4 of [Seg-83]) , the product $C_G(g_1)G$ is closed. Taking into account that $\overline{C_L(g_1)} = C_{\widehat{L}}(g_1)$ by Proposition 3.2, we obtain the equality $\hat{c}\hat{g} = l \in C_{\widehat{L}}(g_1)\widehat{G} \cap L = GC_L(g_1)$. It follows that there exist $g \in G$ and $c \in C_L(g_1)$ such that $cg = l$. It follows that $g_1^g = g_2$ as required. $\square$

Remark 3.6. *By Theorem 4.2 in [BHMS-07] a residually free group of type FP_2 is finitely presented.*

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