Homological invariants for pro-p groups and some finitely presented pro-$\mathcal{C}$ groups

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Abstract

Let G be a finitely presented pro-$\mathcal{C}$ group with discrete relations. We prove that the kernel of an epimorphism of G to $\hat{\mathbb{Z}}_{\mathcal{C}}$ is topologically finitely generated if G does not contain a free pro-$\mathcal{C}$ group of rank 2. In the case of pro-p groups the result is due to J.Wilson and E. Zelmanov and does not require that the relations are discrete [15], [17].

For a pro-p group G of type FP_m we define a homological invariant $\Sigma^m(G)$ and prove that this invariant determines when a subgroup H of G that contains the commutator subgroup G' is itself of type FP_m. This generalises work of J. King for $\Sigma^1(G)$ in the case when G is metabelian [10].

Both parts of the paper are linked via two conjectures for finitely presented pro-p groups G without free non-cyclic pro-p subgroups. The conjectures suggest that the above conditions on G impose some restrictions on $\Sigma^1(G)$ and on the automorphism group of G.

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1 Introduction

This paper contains two parts. The first part is concerned with kernels of surjective homomorphisms $G \to \hat{\mathbb{Z}}_{\mathcal{C}}$, where G is a pro-$\mathcal{C}$ group, $\mathcal{C}$ a class of finite groups closed under subgroups, quotients and extensions and $\hat{\mathbb{Z}}_{\mathcal{C}}$ the pro-$\mathcal{C}$ completion of the infinite cyclic group $\mathbb{Z}$. In the second part we define new homological invariants of pro-p groups. The Conjecture 1 stated at the end of the Introduction can be viewed as a bridge between both parts of the paper.

Theorem 1 *Let G_0 be a discrete finitely presented group and*

$$\varphi_0 \colon G_0 \longrightarrow \mathbb{Z}$$

be an epimorphism. Then the kernel of the induced epimorphism

$$\varphi \colon \widehat{G}_0 \longrightarrow \hat{\mathbb{Z}}_{\mathcal{C}}$$

is either topologically finitely generated or $\widehat{G}_0$ has a free pro-$\mathcal{C}$ group of rank 2, where $\widehat{G}_0$ is the pro-$\mathcal{C}$ completions of G_0.

The proof of Theorem 1 is a modification of a result of R. Bieri and R. Strebel that decomposes a finitely presented discrete group with infinite cyclic quotient as an HNN extension with finitely generated base and finitely generated associated subgroups [2]. In the case when G is a pro-p group (i.e. $\mathcal{C}$ is the class of finite p-groups) the above result is known in more general form i.e. the kernel of every epimorphism φ from a finitely presented pro-p group G to $\mathbb{Z}_p$ for G without non-cyclic pro-p free subgroups is topologically finitely generated. Indeed by a recent result of Zelmanov a pro-p group that satisfies the Golod-Shafarevich type inequality always contains a free non-cyclic pro-p subgroup [17]. Furthermore the proofs of the results of [15, Thm A, Cor A] or their versions in the last chapter of [16] imply that every kernel of a surjective homomorphism of a finitely presented pro-p group that does not satisfy a Golod-Shafarevich type inequality

to $\mathbb{Z}_p$ is (topologically) finitely generated.

In the second part of the paper we consider only pro-p groups. For every $m \geq 1$ we define an invariant $\Sigma^m(G)$ for a pro-p group G of homological type FP_m, FP_1 being (topologically) finitely generated, FP_2 - finitely presented in the category of pro-p groups. By definition

$$T(G) = \{\chi : G \to K[[t]]^{\times} \mid \chi \text{ is continuous homomorphism}\},$$

where K is the algebraic closure of the field with p elements. Define $\overline{\chi}$ as the unique continuous ring homomorphism $\mathbb{Z}_p[[G]] \to K[[t]]$ that lifts χ. We view the (continuous) homology groups $H_i([G,G],\mathbb{Z}_p)$ as (right) pro-p $\mathbb{Z}_p[[G]]$-modules via the right action of G on $[G,G]$ by conjugation. Note that since G is (topologically) finitely generated $[G,G]$ is a closed subgroup of G [13, Lemma 4.2.3]. Observe that $H_i([G,G],\mathbb{Z}_p)$ is in fact a (right) pro-p $\mathbb{Z}_p[[G/[G,G]\ ]]$-module.

Definition 1 *Let m be a natural number and G be a pro-p group of type FP_m. We define $\Sigma^m(G)$ as the set*

$$\{\chi \in T(G)\backslash\{1\} \mid ann_{\mathbb{Z}_p[[G]]} H_i([G,G],\mathbb{Z}_p) \not\subseteq Ker\ \overline{\chi} \text{ for every } 1 \leq i \leq m\},$$

where $ann_{\mathbb{Z}_p[[G]]} H_i([G,G],\mathbb{Z}_p)$ is the annihilator of $H_i([G,G],\mathbb{Z}_p)$ in $\mathbb{Z}_p[[G]]$ i.e. contains those elements $\lambda \in \mathbb{Z}_p[[G]]$ that act trivially on $H_i([G,G],\mathbb{Z}_p)$.

We note that the definition of $\Sigma^m(G)$ makes sense for groups that are not of type FP_m. Still we want to restrict ourselves to the class of groups G of type FP_m as in this case by Lemma 1 from the preliminaries the homology group $H_i([G,G],\mathbb{Z}_p)$ is finitely generated as a pro-p $\mathbb{Z}_p[[G/[G,G]\ ]]$-module for all $i \leq m$. It is obvious by the definition that

$$\ldots \subseteq \Sigma^m(G) \subseteq \Sigma^{m-1}(G) \subseteq \ldots \subseteq \Sigma^1(G) \subseteq T(G).$$

By now the invariant $\Sigma^m(G)$ has been known (in a slightly different form) only in the case $m = 1$, G metabelian and has turned out to be extremely useful in the classification of the metabelian pro-p groups of homological type FP_m [9], [11]. We prove two main results about the new invariants. The first is a pro-p version of a result of R. Bieri and B. Renz for discrete groups [1]. Compared to the discrete case, the proof of the pro-p case is much easier and does not have any geometric flavour, which is hardly a surprise.

Theorem 2 *Let m be a natural number, G be a pro-p group of type FP_m and H be a closed subgroup of G that contains the commutator subgroup $[G, G]$. Then H is of type FP_m if and only if*

$$T(G, H) = \{\chi \in T(G) \mid \chi(H) = 1\} \subseteq \Sigma^m(G) \cup \{1\}.$$

In Lemma 5 we show a characterisation of the invariant $\Sigma^m(G)$ in terms of J. King's invariant $\Delta_V(Q)$ in the special case when G is an extension of V by Q, V and Q are abelian and we view V as a right $\mathbb{Z}_p[[Q]]$-module via conjugation. By definition King's invariant is

$$\Delta_V(Q) = \{\chi \in T(Q) \mid ann_{\mathbb{Z}_p[[Q]]} V \subseteq Ker\ \overline{\chi}\} \cup \{1\}.$$

This invariant turns out to be important for the classification of the metabelian pro-p groups of type FP_m. In Theorem 3 we find another characterisation of the invariant $\Delta_V(Q)$.

Theorem 3 *Let Q be a finitely generated abelian pro-p group, V a (right) finitely generated pro-p $\mathbb{Z}_p[[Q]]$–module. Then $\chi \in T(Q) \setminus \{1\}$ is an element of $\Delta_V(Q)$ if and only if there is a non-zero continuous $\mathbb{F}_p$-linear map*

$$w_\chi \colon V/pV \to K[[t]]$$

such that

$$w_\chi(vq) = w_\chi(v)\chi(q) \text{ for all } v \in V/pV, q \in Q.$$

The proof of Theorem 3 relies on developing commutative algebra methods for the ring $\mathbb{Z}_p[[Q]]$, where Q is a finitely generated abelian pro-p group. Commutative algebra methods have already been helpful in showing a similar result for the Bryant-Groves invariant of metabelian Lie algebras [12, Lemma 2]. It is interesting to note that by now higher dimensional invariants of Lie algebras have not been defined due to the lack of geometric methods. The invariants introduced in this paper cannot be modified for Lie algebras as in the Lie case the homology groups are not entirely responsible for detecting the homological property FP_m. Furthermore in the Lie case it is already known that for a finitely presented Lie algebra L without free non-abelian Lie subalgebras every ideal of codimension one is finitely generated as a Lie algebra [14]. This could be viewed as Lie algebra counterpart of Theorem 1 (without any restrictions about relations). It is worth noting that in [14] only the case of a soluble L algebra is stated but in fact the proof uses only that L has no non-cyclic free Lie subalgebras.

As a corollary of Lemma 5 and Theorem 3 we obtain the following result.

Corollary 1 *Let G be a pro-p group of type FP_m. Then $\chi \in T(G) \setminus \{1\}$ is not in $\Sigma^m(G)$ if and only if there is at least one i such that $1 \leq i \leq m$ and there is a non-zero continuous $\mathbb{F}_p$-linear map*

$$w_{\chi,i}: \mathbb{F}_p \otimes_{\mathbb{Z}_p} H_i([G,G], \mathbb{Z}_p) \to K[[t]]$$

such that for all $v \in \mathbb{F}_p \otimes_{\mathbb{Z}_p} H_i([G,G], \mathbb{Z}_p), q \in Q = G/[G,G]$,

$$w_{\chi,i}(vq) = w_{\chi,i}(v)\chi(q),$$

where Q acts on the homology group $H_i([G,G], \mathbb{Z}_p)$ via the right action of G on the commutator $[G,G]$ by conjugation.

We state two conjectures, the first is motivated by the main result of [3]. We call two elements χ_1, χ_2 of $T(G)$ antipodal if $\chi_1 \chi_2 = 1$. An element χ of $T(G)$ is trivial if $\chi = 1$.

Conjecture 1 *Let G be a pro-p group such that G is finitely presented (in the pro-p sense) and $[G,G]$ is without non-cyclic free pro-p subgroups. Then $T(G) \setminus \Sigma^1(G)$ does not have non-trivial antipodal elements.*

Conjecture 2 *Let G be a pro-p group such that G is finitely presented (in pro-p sense), $[G,G]$ is not (topologically) finitely generated and $[G,G]$ is without non-cyclic free pro-p subgroups. Then there does not exist a (continuous) automorphism β of G such that the induced by β automorphism to $Q = G/[G,G]$ is the antipodal automorphism.*

Note that Conjecture 1 if true would imply Conjecture 2. As well it would imply another proof of the more general version of Theorem 1 in the pro-p case (i.e. without any restrictions on the type of relations) that does not use Zelmanov's result [17]. Indeed if G is a pro-p group satisfying the assumptions of Conjecture 1 then by the conclusion of Conjecture 1 and the King's criterion [9] $G/\overline{G''}$ is finitely presented (in the pro-p sense). Finally by [15, Cor A] any kernel of a projection of a finitely presented soluble pro-p group to $\mathbb{Z}_p$ is (topologically) finitely generated.

Furthermore by King's classification [9] Conjecture 1 holds for metabelian pro-p groups. Hence Conjecture 2 holds for metabelian pro-p groups. We show in Proposition 1 that the property deduced in Conjecture 2 does not characterise finite presentability of metabelian pro-p groups, namely there is a (topologically) finitely generated metabelian pro-p group that is not finitely presented (in the pro-p sense) and there is no (continuous) automorphism of G that induces the antipodal map on the abelianization $G/[G,G]$. In contrast the property deduced in Conjecture 1 characterises finite presentability in the category of metabelian pro-p groups.

2 Preliminaries

2.1 Homological properties of pro-p groups

In this section G is a pro-p group, $\mathbb{Z}_p[[G]]$ is the completed group algebra of G i.e. the inverse limit of ordinary group algebras

$$(\mathbb{Z}/p^k\mathbb{Z})[G/N]$$

over all $k \in \mathbb{N}$ and all open normal subgroups N of G. We call a (right) module M over $\mathbb{Z}_p[[G]]$ a pro-p $\mathbb{Z}_p[[G]]$-module if its additive structure makes it a pro-p group, the action of $\mathbb{Z}_p[[G]]$ on M is continuous and the action of $\mathbb{Z}_p$ is the standard one. The category of pro-p $\mathbb{Z}_p[[G]]$-modules has enough projectivities i.e. every module is a quotient of a free module and this allows us to develop a homological machinery. A good reference about homological properties of pro-finite groups is [13].

A pro-p group G is said to be of homological type FP_m if there is a projective (continuous) resolution of the trivial pro-p $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p$

$$\mathcal{F}\colon \ldots \to F_i \to F_{i-1} \to \ldots \to F_0 \to \mathbb{Z}_p \to 0$$

with all F_i (topologically) finitely generated pro-p $\mathbb{Z}_p[[G]]$-modules for $i \leq m$. It is not difficult to see that G is of type FP_m if and only if all continuous homology groups $H_i(G, \mathbb{F}_p) = Tor_i^{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, \mathbb{F}_p)$ are finite for $i \leq m$, where $Tor_i^{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, V)$ for V a (left) pro-p $\mathbb{Z}_p[[G]]$-module is the homology group in the category of pro-p groups i.e. it is calculated using a continuous projective resolution of $\mathbb{Z}_p$. In fact the equivalence between G having type FP_m and $H_i(G, \mathbb{F}_p)$ being finite for $i \leq m$ is an easy corollary of the fact that a subset X of a pro-p $\mathbb{Z}_p[[G]]$-module M is a (topological) generating set if and only if the image of X in $M \otimes_{\mathbb{Z}_p[[G]]} \mathbb{F}_p$ is a (topological) generating set over $\mathbb{F}_p$. Furthermore if X is finite and generates M as pro-p $\mathbb{Z}_p[[G]]$-module then it generates M as abstract module over the ring $\mathbb{Z}_p[[G]]$ [5, 1.5].

The above criterion for groups G of type FP_m can be slightly improved when G has a normal subgroup N such that $\mathbb{Z}_p[[G/N]]$ is topologically (right) Noetherian i.e. increasing sequences of closed submodules always stabilize. Note that the completed group algebras of pro-p groups of finite rank (in the sense of [7]) are topologically (right) Noetherian [16, Thm 8.7.8].

Lemma 1 ([10]) *Suppose G is a pro-p group with normal closed subgroup N such that G/N has finite rank. Then G is of type FP_m if and only if $H_i(N, \mathbb{F}_p)$ is a finitely generated (right) pro-p $\mathbb{F}_p[[G/N]]$-module (topologically or abstractly is the same) for all $i \leq m$, where the right action of G/N on $H_i(N, \mathbb{F}_p)$ is induced by the conjugation of G on N.*

2.2 King's invariant for metabelian groups

Suppose Q is a finitely generated abelian pro-p group and A is a finitely generated (right) pro-p $\mathbb{Z}_p[[Q]]$-module. King's invariant

$$\Delta_A(Q) = \{\chi \in T(Q) \mid ann_{\mathbb{Z}_p[[Q]]} A \subseteq Ker\ \overline{\chi}\} \cup \{1\}$$

turns out to be important for the classification of the metabelian pro-p groups of type FP_m stated in the following theorem. The case $m = 2$ is done in [9] and the case of general m can be found in [11].

Theorem 4 *Let $1 \to A \to G \to Q \to 1$ be a short exact sequence of pro-p groups, G (topologically) finitely generated, A and Q abelian and $m > 1$ be an integer. Then the following are equivalent :*

1. G is of type FP_m;

2. The m-th completed exterior power of A is finitely generated over $\mathbb{Z}_p[[Q]]$ via the diagonal Q-action;

3. The m-th completed tensor power of A is finitely generated over $\mathbb{Z}_p[[Q]]$ via the diagonal Q-action;

4. Whenever $\chi_1, \ldots, \chi_m \in \Delta_A(Q)$ and $\chi_1 \ldots \chi_m = 1$ then all χ_i are 1.

Note that for G metabelian, $A = [G, G]$ and $Q = G/A$ we have that $H_1(A, \mathbb{Z}_p) \simeq A$ and hence there is a natural isomorphism between $\Delta_A(Q)$ and $\Sigma^1(G)^c := T(G) \backslash \Sigma^1(G)$. Furthermore the invariant $\Delta_A(Q)$ is important for understanding the structure of the module A.

Theorem 5 ([9]) *Suppose Q is a (topologically) finitely generated abelian pro-p group, H a (closed) subgroup of Q and A a finitely generated pro-p $\mathbb{Z}_p[[Q]]$-module. Then A is finitely generated as a pro-p $\mathbb{Z}_p[[H]]$-module if and only if*

$$T(Q, H) \cap \Delta_A(Q) = \{1\},$$

where $T(Q, H) = \{\chi \in T(Q) \mid \chi(H) = 1\}$. In particular A is (topologically) finitely generated if and only if $\Delta_A(Q) = \{1\}$.

Finally we remind the reader that the invariant $\Delta_A(Q)$ is additive.

Lemma 2 [9, Lemma 2.3] *Let $0 \to A_1 \to A \to A_2 \to 0$ be a short exact sequence of finitely generated pro-p $\mathbb{Z}_p[[Q]]$-modules. Then*

$$\Delta_A(Q) = \Delta_{A_1}(Q) \cup \Delta_{A_2}(Q).$$

3 Proof of Theorem 1

The proof of Theorem 1 uses the construction of pro-$\mathcal{C}$ HNN extension. Following [13, p. 390] we consider a pro-$\mathcal{C}$ group H with closed subgroups A and B and with a continuous isomorphism $\eta : A \to B$. Then the pro-$\mathcal{C}$ HNN-extension $HNN_{\mathcal{C}}(H, A, B, t)$ with base H, associated subgroups A and B and stable letter t is a pro-$\mathcal{C}$ group given by presentation $\langle H, t \mid t^{-1}at = \eta(a), a \in A\rangle$. It satisfies a universal property as in the discrete case and there is a natural homomorphism of pro-$\mathcal{C}$ groups from H to $HNN_{\mathcal{C}}(H, A, B, t)$ that in general is not injective. When this map is injective we say that H is embedded in $HNN_{\mathcal{C}}(H, A, B, t)$. The images of A and B in $HNN_{\mathcal{C}}(H, A, B, t)$ are isomorphic via conjugation with t.

Lemma 3 *Let $G = HNN_{\mathcal{C}}(H, A, B, t)$ be a pro-$\mathcal{C}$ HNN-extension with the base group H embedded in G and associated subgroups A,B. If G does not contain non-abelian free pro-$\mathcal{C}$ subgroups then $A = B = H$.*

Proof Suppose without loss of generality $A \neq H$. Then there exists an open normal subgroup N of G such that $NA \neq NH$. Note that by the universal property of pro-$\mathcal{C}$ HNN-extensions [13, p. 390] the canonical homomorphism $H \to HN/N$ extends to a homomorphism of pro-$\mathcal{C}$ groups

$$G = HNN_{\mathcal{C}}(H, A, B, t) \to HNN_{\mathcal{C}}(HN/N, AN/N, BN/N, t),$$

that is in fact an epimorphism. Thus we may assume that H, A, B are in $\mathcal{C}$, hence finite. Then G is just the pro-$\mathcal{C}$ completion $K_{\mathcal{C}}$ of the discrete HNN-extension K of H with associated subgroups A, B. Since H is finite and $A \neq H$ we have $|B| = |A| < |H|$, hence $B \neq H$.

Let $\theta : K \to G = K_{\mathcal{C}}$ be the canonical homomorphism from a group to its pro-$\mathcal{C}$ completion. As H embeds in G there is an epimorphism $\psi : G \to L$, where $L \in \mathcal{C}$ such that $H \cap Ker\ \psi = 1$. Let F be the kernel of the homomorphism $\varphi = \psi\theta : K \to L$. By [6, Section 8.5, Thm 27] F is the fundamental group of a connected graph of trivial groups, say with α vertices and β edges, where α is the number of double coset classes FgH in G and β is the number of double coset classes FgA in G. If s is the index of A in H then $\beta = s\alpha$. As a maximal subtree in a connected graph with α vertices has $\alpha - 1$ edges it follows that F is free of rank $\beta - (\alpha - 1) = \alpha(s-1) + 1 \geq \alpha + 1 \geq 2$ i.e. F is non-abelian.

Note that K/F is a subgroup of L, $\mathcal{C}$ is subgroup closed and hence $K/F \in \mathcal{C}$. Then the fact that $\mathcal{C}$ is extension closed implies that the pro-$\mathcal{C}$ topology of K induces on F the full pro-$\mathcal{C}$ topology. Finally by [13, Lemma 3.2.6] the pro-$\mathcal{C}$ completion of F embeds in $G = K_{\mathcal{C}}$. □

Proof of Theorem 1.

Let F be a finitely generated free discrete group such that G_0 is a quotient of F, $\pi\colon F \to G_0$ be the canonical epimorphism and the kernel of π be a normal subgroup generated by a finite subset R of F. Then one has the induced surjective homomorphism

$$\psi\colon \widehat{F_{\mathcal{C}}} \to G = \hat{G}_0$$

with kernel the minimal normal closed subgroup containing R, where $\widehat{F_{\mathcal{C}}}$ is the pro-$\mathcal{C}$ completion of F. Without loss of generality we can assume that F has a basis $X = \{x_1, \ldots, x_n\}$, $\varphi_0\pi(x_1)$ is a generator of $\mathbb{Z}$ and $\varphi_0\pi(x_i) = 0$ for all $i \geq 2$. Let F_1 be the free subgroup of F with basis $X_1 = \{x_i^{x_1^j}\}_{2 \leq i \leq n, j \in \mathbb{Z}}$, thus $F_1 = Ker(\varphi_0\pi)$, and $\overline{F_1}$ be the closure of F_1 in $\widehat{F_{\mathcal{C}}}$. Then for $N = Ker\ \varphi$ we have $\psi^{-1}(N) = \overline{F_1} \subset \widehat{F_{\mathcal{C}}}$ and all relators R are in $\psi^{-1}(N)$. Note that there is some natural number d such that R is in the subgroup of F_1 generated by $\{x_i^{x_1^j}\}_{2 \leq i \leq n, -d \leq j \leq d}$.

Now we construct a pro-$\mathcal{C}$ HNN extension $\widetilde{G}$ that will turn out to be isomorphic to G. First we define the subgroups A, B and D of G as the subgroups that are (topologically) generated by

$$\{\psi(x_i^{x_1^j})\}_{i \geq 2, -d \leq j \leq d-1}, \{\psi(x_i^{x_1^j})\}_{i \geq 2, -d+1 \leq j \leq d}$$

$$\text{and } \{\psi(x_i^{x_1^j})\}_{i \geq 2, -d \leq j \leq d} \text{ respectively.}$$

Then $\widetilde{G}$ is given by the pro-$\mathcal{C}$ presentation $\langle D, t \mid A^t \simeq B\rangle$, where the isomorphism $A^t \simeq B$ is given by $t^{-1}at = \psi(x_1)^{-1}a\psi(x_1) \in B$. We claim that D embeds in $\widetilde{G}$ via the above presentation of $\widetilde{G}$. In general the base of a pro-$\mathcal{C}$ group does not embed in pro-$\mathcal{C}$ HNN extension [8] but in our case D is already a subgroup of G i.e. we can construct first a homomorphism from the free pro-$\mathcal{C}$ product $D * \hat{\mathbb{Z}}_C$ to G which is identity on D and sends a generator t of $\hat{\mathbb{Z}}_C$ to $\psi(x_1)$ and then factor it through $\widetilde{G}$ to get a homomorphism of pro-$\mathcal{C}$ groups

$$\mu: \widetilde{G} \to G.$$

We show that μ is an isomorphism by constructing its inverse. Let θ be the homomorphism of pro-$\mathcal{C}$ groups $\widehat{F_{\mathcal{C}}} \to \widetilde{G}$ sending x_1 to t, x_i to $\psi(x_i) \in D$ for $i \geq 2$. We claim that the choice of D implies that $\theta(R) = 1$, hence θ factors through G and gives the inverse of μ. Indeed an element $r \in R$ is a word $w = w(x_i^{x_1^j}: i \geq 2, -d \leq j \leq d)$ and $\theta(w) = w(\psi(x_i)^{t^j}: i \geq 2, -d \leq j \leq d)$. Note that by the definition of the base and associated subgroups of $\widetilde{G}$ we have $\psi(x_i)^{t^j} = \psi(x_i^{x_1^j}) \in D \subset \widetilde{G}$ for $i \geq 2, -d \leq j \leq d$. Thus $\theta(w) = w(\psi(x_i^{x_1^j}): i \geq 2, -d \leq j \leq d) \in D$ and $w(\psi(x_i^{x_1^j}): i \geq 2, -d \leq j \leq d) = \psi(w(x_i^{x_1^j}: i \geq 2, -d \leq j \leq d)) = \psi(r) = 1$ as required.

Now we have decomposed G as a pro-$\mathcal{C}$ HNN extension with (topologically) finitely generated base D and (topologically) finitely generated associated subgroups A and B. By Lemma 3 a pro-$\mathcal{C}$ HNN extension where the base embeds in the group and is not equal to the associated subgroups always contains a free pro-$\mathcal{C}$ subgroup of rank 2. If G does not contain such free pro-$\mathcal{C}$ subgroup it follows that $D = A = B = A^{\psi(x_1)}$. Hence $A = D$ is normal in G and equal to the kernel of φ. This completes the proof of the theorem as D by definition is (topologically) finitely generated.□

Corollary 2 *Let G be a finitely presented pro-$\mathcal{C}$ group with discrete relations R and without non-cyclic free pro-$\mathcal{C}$ subgroups. Suppose G admits an infinite procyclic quotient. Then $G = N \rtimes \hat{\mathbb{Z}}_{\mathcal{C}}$ with N (topologically) finitely generated.*

Proof Let $\mu: G = \hat{G}_0 \to M$ be an epimorphism on an infinite procyclic quotient M of G and $\beta: G_0 \to \hat{G}_0$ be the canonical map. As $\beta(G_0)$ is dense in G and M is infinite and abelian we deduce that the abelianization of $\beta(G_0)$ is infinite, hence $G_0/[G_0, G_0]$ is infinite. Then as G_0 is finitely generated it has $\mathbb{Z}$ as an epimorphic image. Finally we apply Theorem 1. $\square$

Note that the fact that $\mathcal{C}$ is extension closed was used only in the last paragraph of the proof of Lemma 3. Therefore, one obtains the same results making the restriction on $\mathcal{C}$ milder, but strengthening the restriction on G.

Lemma 4 *Let $\mathcal{C}$ be a class of finite groups closed for subgroups, homomorphic images and extensions with abelian kernel. Let*

$$G = HNN_{\ \mathcal{C}}(H, A, B, t)$$

be a pro-$\mathcal{C}$ HNN-extension with the base group H embedded in G and associated subgroups A,B. If G does not contain non-abelian free pro-p subgroups for any prime p then $A = B = H$.

Proof It suffices to show that a pro-$\mathcal{C}$ HNN extension $G = HNN_{\ \mathcal{C}}$ (H, A, B, t) where the base embeds in the group and is not equal to the associated subgroups always contains a free pro-p subgroup of rank 2. Suppose without loss of generality that $A \neq H$. Then there exists an open normal subgroup N of G such that $NA \neq NH$. Note that $G_1 = HNN_{\ \mathcal{C}}(HN/N, AN/N, BN/N, t)$ is a quotient of G. By [13, Prop. 7.6.7] a free pro-p group is projective as a profinite group, hence if G_1 contains a free pro-p subgroup then there is an isomorphic copy of this subgroup in G. Thus from now on we may assume that H, A, B are in $\mathcal{C}$, hence are finite.

Let $\mathcal{C}_0$ be the smallest class of finite groups that contains $\mathcal{C}$ and is closed for taking subgroup, quotient and extensions. Let $M = HNN_{\mathcal{C}_0}(H, A, B, t)$ be the pro-$\mathcal{C}_0$ HNN-extension. Since H embeds in G the kernel K_0 of the natural epimorphism of M to G intersects H trivially. So there exists an open subgroup F of M containing K_0 that intersects H trivially and therefore F is a free non-abelian pro-$\mathcal{C}_0$ group (see the proof of Lemma 3).

Since $\mathcal{C}$ is closed under extensions with abelian kernel, K_0 is perfect. Therefore, K_0 is in the kernel of an epimorphism of F onto its maximal pro-p quotient F_p for every p. Furthermore as F is non-abelian for some prime q the free group F_q is non-abelian. Thus F_q is a quotient of F/K_0 and hence embeds in F/K_0 and in G. $\square$

Theorem 6 *Let $\mathcal{C}$ be a class of finite groups closed for subgroups, homomorphic images and extensions with abelian kernel. Let $G = \langle x_1, x_2, \ldots, x_n \mid R\rangle$ be a finitely presented pro-$\mathcal{C}$ group having discrete relations R and having no non-cyclic free pro-p subgroups for every prime p. Then the following hold:*

(i) If $\varphi: G \to \hat{\mathbb{Z}}_{\mathcal{C}}$ is an epimorphism induced by an epimorphism of discrete groups $G_0 = \langle x_1, \ldots, x_n \mid R\rangle \longrightarrow \mathbb{Z}$, then the kernel of φ is topologically finitely generated ;

(ii) Under the assumptions of (i) $G = N \rtimes \hat{\mathbb{Z}}_{\mathcal{C}}$ for some (topologically) finitely generated normal subgroup N of G.

Proof. The proof is the same as the proof of Theorem 1 except that we use Lemma 4 instead of Lemma 3. $\square$

4 Applications of commutative algebra methods to pro-p groups

We start this section with a simple lemma that is a straight corollary of the definitions of $\Sigma^m(G)$ and King's invariant $\Delta_A(Q)$.

Lemma 5 *Let m be a natural number and G be a pro-p group of type FP_m. Then the canonical projection $G \to G/[G,G]$ sends*

$$\cup_{1 \leq i \leq m} \Delta_{H_i([G,G],\mathbb{Z}_p)}(G/[G,G])$$

bijectively to $T(G) \setminus \Sigma^m(G)$.

Proof of Theorem 2

We note first that $H/[G,G]$ is a (topologically) finitely generated abelian pro-p group and hence we can apply Lemma 1 i.e. H is of type FP_m if and only if $H_i([G,G],\mathbb{Z}_p)$ is finitely generated over $\mathbb{Z}_p[[H/[G,G]\ ]]$ via the right action of H on $[G,G]$ by conjugation for all $i \leq m$. Note that as G is of type FP_m by Lemma 1 $H_i([G,G],\mathbb{Z}_p)$ is finitely generated over $\mathbb{Z}_p[[G/[G,G]\ ]]$ for every $i \leq m$. By Theorem 5 $H_i([G,G],\mathbb{Z}_p)$ is finitely generated over $\mathbb{Z}_p[[H/[G,G]\ ]]$ if and only if $T(G/[G,G],H/[G,G]) \cap \Delta_{H_i([G,G],\mathbb{Z}_p)}(G/[G,G]) = 1$. This together with Lemma 5 and the natural isomorphism $T(G,H) \simeq T(G/[G,G],H/[G,G])$ completes the proof of the theorem.

Proof of Theorem 3.

1) First we show that the existence of w_χ implies that $\chi \in \Delta_V(Q)$. Let $\overline{v} \in V/pV$ such that $w_\chi(\overline{v}) \neq 0$ and $I = ann_{\mathbb{Z}_p[[Q]]}(V)$. Then $0 = w_\chi(0) = w_\chi(\overline{v}I) = w_\chi(\overline{v})\overline{\chi}(I)$ and since $K[[t]]$ does not have zero-divisors $\overline{\chi}(I) = 0$.

2) The difficult part of the proof is that $\chi \in \Delta_V(Q)$ implies the existence of w_χ. The starting point of our considerations is a lemma by J.Wilson that for a profinite ring R and a profinite R-module M every abstractly finitely generated R-submodule of M is a closed submodule [16, Lemma 7.2.2]. In our case $R = \mathbb{Z}_p[[Q]]$, where Q is a (topologically) finitely generated abelian pro-p group. As $\mathbb{Z}_p$ is a principal ideal domain and $\mathbb{Z}_p[[Q]] \simeq \mathbb{Z}_p[[t_1,\dots,t_m]] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[Q_{tor}]$, where m is the torsion-free rank of Q and Q_{tor} is the torsion part of Q, we see that $\mathbb{Z}_p[[Q]]$ is abstractly Noetherian i.e. every ideal is (abstractly) finitely generated as $\mathbb{Z}_p[[Q]]$-module, hence is closed. In particular we can use the commutative algebra methods developed for abstract Noetherian rings (warning: the rings $\mathbb{Z}_p[[Q]]$ and $\mathbb{F}_p[[Q]]$ are not abstractly finitely generated as rings over the basic ring $\mathbb{Z}_p$ and the field $\mathbb{F}_p$, respectively).

From now on we assume that the theorem is true for every proper quotient of V but not for V itself. Now let $\chi \neq 1$ be an element of $\Delta_V(Q)$. We write P for the kernel of the extension $\overline{\chi}\colon \mathbb{Z}_p[[Q]] \to K[[t]]$ of χ and observe that if $\chi \in \Delta_W(Q)$ for a proper quotient W, then the assumption allows us to find a map $w: W \to K[[t]]$ which can then be extended to V via the natural epimorphism $V \to W$ and we are done. Then we can assume that $\chi \notin \Delta_W(Q)$ for any proper quotient W of V. Thus by Lemma 2, $\chi \in \Delta_{W_1}(Q)$ for any non-zero submodule W_1 of V (in this case $W = V/W_1$).

Consider all associative primes $P_1, \ldots, P_s$ of V as $\mathbb{Z}_p[[Q]]$-module. Note that we have only finitely many associative primes as V is a finitely generated (topologically or abstractly is the same) $\mathbb{Z}_p[[Q]]$-module and $\mathbb{Z}_p[[Q]]$ is abstractly Noetherian. Thus for every associative prime P_i there is an element $v_i \in V$ such that $P_i = Ann_{\mathbb{Z}_p[[Q]]}(v_i)$. Then for the non-zero submodule $W_1 = \mathbb{Z}_p[[Q]]v_i \simeq \mathbb{Z}_p[[Q]]/P_i$ of V we have that $\chi \in \Delta_{W_1}(Q)$, hence $P_i \subseteq Ker(\overline{\chi}) = P$ i.e. any associated prime of V is in P.

Now we consider the primary decomposition $0 = \cap_{i \leq s} L_i$ of the trivial submodule of V given by [4, Ch. 4, Section 2, Thm 1] i.e. all quotients V/L_i are $\mathbb{Z}_p[[Q]]$-modules with one associated prime depending on i. If V has more than one associated prime then all L_i are non-zero, and hence V/L_i are proper quotients of V. Note that V embeds in $V/L_1 \oplus \ldots \oplus V/L_s$ and by additivity of Δ given by Lemma 2

$$\chi \in \Delta_V(Q) \subseteq \Delta_{V/L_1 \oplus \ldots \oplus V/L_s}(Q) = \cup_{1 \leq i \leq s} \Delta_{V/L_i}(Q)$$

i.e. for some i we have $\chi \in \Delta_{V/L_i}(Q)$, a contradiction. Therefore we are in the case when V has just one associated prime I (remember every associated prime is in P, hence $I \subseteq P$). Hence $V.I^m = 0$ for some m and $V/V.I$ has annihilator I in $\mathbb{Z}_p[[Q]]$. If $V.I \neq 0$ then $V/V.I$ is a proper quotient of V with annihilator $I \subseteq P$ and so $\chi \in \Delta_{V/V.I}(Q)$, a contradiction. In other words we can assume that $V.I = 0$ i.e. I is the annihilator of V in $\mathbb{Z}_p[[Q]]$. Furthermore the maximal elements in the set $\{Ann(x) \mid x \in V \setminus \{0\}\}$ are associated primes for V [4, Ch. 4, Section 1, Prop. 2], thus for every $x \in V \setminus \{0\}$ its annihilator in $\mathbb{Z}_p[[Q]]$ is the ideal I. All this shows that V is torsion-free as $\mathbb{Z}_p[[Q]]/I$–module.

Now we aim to prove that the annihilator of $V/V.P$ in $\mathbb{Z}_p[[Q]]$ is P. This will imply that if $I \neq P$ the module $V/V.P$ is a proper quotient of V and as $P = Ker(\overline{\chi})$ we have $\chi \in \Delta_{V/V.P}(Q)$, a contradiction. Now suppose that the annihilator of $V/V.P$ is bigger than P, i.e.

contains an element $\lambda \in \mathbb{Z}_p[[Q]] \setminus P$. Then for a finite generating set $e_1, \dots, e_m$ of V over $\mathbb{Z}_p[[Q]]$ there exist elements $f_{i,j} \in P$ such that $e_i\lambda = \sum_j e_j f_{i,j}$ for all $i \leq m$. Then $e_i det(A) = 0$ for every $i \leq m$, where A is the matrix with entries $\lambda\delta_{i,j} - f_{i,j}$, $\delta_{i,j}$ is the Kronecker symbol, and hence $det(A) \in ann(e_j) = I \subseteq P$. As $det(A) \in \lambda^m + P$ we have that $\lambda^m \in P$, a contradiction as P is a prime ideal and $\lambda \notin P$.

From now on we assume $I = P$. We claim that there is a suitable $\mathbb{F}_p$-linear map $w \colon V \to K((t))$ extending χ. First we show how the existence of such a map completes the proof of the theorem. As V is finitely generated over $\mathbb{Z}_p[[Q]]$ we have that $w(V)$ is finitely generated over $\overline{\chi}(\mathbb{Z}_p[[Q]]) \subset K[[t]]$ and hence $Im\ w \subseteq t^{-m}K[[t]]$ for some $m \in \mathbb{N}$. Now applying a multiplication with t^m after w we obtain a map $V \to K[[t]]$ extending χ, which will be a contradiction completing the proof of the theorem.

Finally we show the existence of the map w. Let $R = \mathbb{Z}_p[[Q]]/P$ and let S be the field of fractions of R. Observe that χ induces an injective homomorphism χ' from S to $K((t))$. As V is R-torsion-free, it embeds in $V \otimes_R S$ which is a finite dimensional vector space over S and has one dimensional quotient S that embeds in $K((t))$ via χ'. The composition of all these maps gives the desired map w and thus completes the proof.$\square$

Proposition 1 *There exists a (topologically) finitely generated metabelian pro-p group G such that*

(i) G is not finitely presented (in the pro-p sense);

(ii) $G/[G,G] \simeq \mathbb{Z}_p^2$;

(iii) there does not exist a (continuous) automorphism β of G such that the automorphism induced by β on $Q = G/[G,G]$ is the antipodal automorphism.

Proof Let α be the (continuous) ring automorphism $\mathbb{F}_p[[Q]]$ that sends $q \in Q$ to q^{-1}, where $Q = G/G' \simeq \mathbb{Z}_p^n$, Q acts on $A = G'$ via right conjugation and A is of exponent p. Furthermore assume that A is cyclic $\mathbb{F}_p[[Q]]$-module with generator a and the annihilator of A

in $\mathbb{F}_p[[Q]]$ is an ideal I i.e. $A \simeq \mathbb{F}_p[[Q]]/I$. Furthermore we assume that $\alpha(I) \neq I$ and $\alpha(P) = P$ for at least one prime ideal P in $\mathbb{F}_p[[Q]]$ with infinite index in $\mathbb{F}_p[[Q]]$ and such that $I \subseteq P$. Note that $A_1 = \mathbb{F}_p[[Q]]/P$ is a quotient of A that is not (topologically) finitely generated as an additive pro-p group and hence $\Delta_{A_1}(Q) \neq \{1\}$. Let $\chi_1 \in \Delta_{A_1}(Q) \setminus \{1\}$. Then as $\alpha(P) = P$ we have that $\chi_2 = \chi_1(\alpha\mid_Q) \in \Delta_{A_1}(Q)$, hence $\chi_1\chi_2 = 1$. As $\Delta_{A_1}(Q) \subseteq \Delta_A(Q)$ we get that $\Delta_A(Q)$ has antipodal non-trivial elements, hence G is not finitely presented as a pro-p group.

We show that as $\alpha(I) \neq I$ there does not exist an automorphism β that induces antipodal maps on the abelianization of G. Assume first that such β exists. Now considering the restriction of β on $A \simeq \mathbb{F}_p[[Q]]/I$ and writing overline for the canonical homomorphism $\mathbb{Z}_p[Q] \to \mathbb{Z}_p[Q]/I$ we have $\beta(\overline{q}) = \beta(\overline{1})\overline{q^{-1}} = \overline{\lambda_1 q^{-1}}$, for some fixed $\lambda_1 \in \mathbb{F}_p[[Q]]$ and any $q \in Q$. As β is an automorphism there exists $\mu_1 \in \mathbb{F}_p[[Q]]$ such that $\overline{\lambda_1\alpha(\mu_1)} = \beta(\overline{\mu_1}) = \overline{1}$. Then $\lambda_1\alpha(\mu_1) \in 1 + I$. Now as $\overline{0} = \beta(\overline{0}) = \beta(I) = \beta(1.I) = \beta(\overline{1})\beta(I) = \overline{\lambda_1\alpha(I)}$ we have $\lambda_1\alpha(I) \subset I$. As λ_1 is invertible mod I we have $\alpha(I) \subseteq I$. Then applying α we have $I = \alpha^2(I) \subseteq \alpha(I)$, a contradiction.

Finally we give an explicit construction in the case $n = 2$. The group G has 3 generators : x, y, a with $x^{-1}y^{-1}xy = a$ and the smallest closed normal subgroup A of G that contains a is abelian and of exponent p. Define P as the ideal of $\mathbb{F}_p[[Q]]$ generated by $x - 1$. As $\alpha(x-1) = x^{-1} - 1 = -x^{-1}(x-1)$ we get $\alpha(P) = P$.

If $p \neq 2$ we define I to be the ideal generated by $w = (x-1)(x+y-2)$. Then $\alpha(w) = (x^{-1}-1)(x^{-1}+y^{-1}-2) = -x^{-2}y^{-1}(x-1)(x+y-2xy)$ and define $w_1 = (x-1)(x+y-2xy)$. Assume that $\alpha(I) = I$, then $2^{-1}(w - w_1) = (x-1)(xy-1) \in I$. Then for some $\lambda \in \mathbb{F}_p[[Q]]$ we have $(x-1)(xy-1) = (x-1)(x+y-2)\lambda$ and hence $xy - 1 = (x+y-2)\lambda$, a contradiction. Thus $\alpha(I) \neq I$.

If $p = 2$ we define I to be the principal ideal generated by $w = (x-1)(x-1+(y-1)^3) = (x-1)(x+y+y^2+y^3)$. Then $\alpha(w) = (x^{-1}-1)(x^{-1}+y^{-1}+y^{-2}+y^{-3}) = -x^{-2}y^{-3}(x-1)(y^3+x+xy+xy^2)$ and define $w_1 = (x-1)(y^3+x+xy+xy^2)$. Suppose that $\alpha(I) = I$. Then $w+w_1 = (x-1)(y+y^2+xy+xy^2) = (x-1)^2(y+y^2) \in I$. In particular there is $\lambda \in \mathbb{F}_2[[Q]]$ such that $(x-1)^2(y+y^2) = (x-1)(x+y+y^2+y^3)\lambda$.

Then $(x-1)(y+y^2) = (x+y+y^2+y^3)\lambda$, hence $\lambda = (x-1)\mu$ for some $\mu \in \mathbb{F}_2[[Q]]$ and $y+y^2 = (x+y+y^2+y^3)\mu$, a contradiction. Thus $\alpha(I) \neq I$. $\square$

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