On Quasifree Profinite Groups

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A recent characterization of free profinite groups due to Harbater and Stevenson [Theorem 2.1, HS] establishes that a profinite group G is free profinite of infinite rank m if and only if

(i) G is projective, and

(ii) whenever one has a diagram

$$\begin{array}{ccc} & & G \\ & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{\alpha} & B \end{array}$$

where A and B are finite groups, α and f are epimorphisms of profinite groups and α splits, there exist exactly m different epimorphisms $\lambda : G \longrightarrow A$ such that $\alpha\lambda = f$.

This builds on other well-known characterizations due to Iwasawa [I], Mel'nikov [M] and Chatzidakis [C] (see [RZ], Theorems 3.5.9 and 3.5.11 for a unified treatment in a slightly more general context).

In this paper we are interested in profinite groups that satisfy condition (ii) above. For an infinite cardinal m, we define a profinite group G to be *m-quasifree* if it satisfies condition (ii) above. The following result of Harbater and Stevenson provides naturally arising examples of m-quasifree groups which are not projective, and hence not free profinite.

Theorem [Theorem 1.1, HS] *Let k be a field and $k((x,t))$ be the fraction field of the power series ring $k[[x,t]]$, where x and t are indeterminates. Let $G = G_{k((x,t))}$ be the absolute Galois group of $k((x,t))$. Denote by m the cardinality of $k((x,t))$. Then G is an m-quasifree profinite group which is not projective.*

In our main result (Theorem 5.1) we show that open subgroups of m-quasifree groups are $\bar{m}$-quasifree. We also provide nonobvious examples of m-quasifree profinite groups.

1. Preliminaries and Examples

Throughout this paper $\mathcal{C}$ denotes a variety of finite groups, i.e., a nonempty class of finite groups closed under the operations of taking subgroups, homomorphic images and finite direct products. For example $\mathcal{C}$ can be taken to be the class of all finite groups or the class of all finite solvable groups. A pro-$\mathcal{C}$ groups is an inverse limit of groups in $\mathcal{C}$. We follow the notation and terminology of [RZ], where basic properties of these groups can be found.

Recall that an epimorphism $\alpha : A \longrightarrow B$ is said to split if there exists a homomorphism $\tau : B \longrightarrow A$ such that $\alpha\tau = \mathrm{id}_B$.

Definition 1.1 *Let $\mathcal{C}$ be a variety of finite groups and let m be an infinite cardinal. A pro-$\mathcal{C}$ group Q is called an m-quasifree pro-$\mathcal{C}$ group if for every diagram of the form*

$$\begin{array}{ccc} & & Q \\ & & \downarrow{\scriptstyle f} \\ A & \overset{\alpha}{\twoheadrightarrow} & B \end{array}$$

where A and B are finite groups in $\mathcal{C}$, α and f are epimorphisms of profinite groups and α splits, there exists exactly m different epimorphisms $\lambda : Q \longrightarrow A$ such that $\alpha\lambda = f$.

We refer to such diagram as a *split embedding problem* of pro-$\mathcal{C}$ groups for Q, and we say that an epimorphism $\lambda : Q \longrightarrow A$ such that $\alpha\lambda = f$ is a *solution* of the embedding problem. Hence Q is m-quasifree if every finite split embedding problem has exactly m different solutions.

Lemma 1.2 *The minimal number of generators $d(Q)$ of an m-quasifree pro-$\mathcal{C}$ group Q is $d(Q) = m$.*

Proof. Recall that the local weight $w_0(Q)$ of an infinite profinite group Q is the number of open normal subgroups of Q. Note that the minimal number of generators $d(Q)$ of Q equals its local weight, $d(Q) = w_0(Q)$, since Q is infinitely generated (see Proposition 2.6.2 in [RZ]). So it suffices to prove that $w_0(Q) = m$. For any open normal subgroup N of Q, the number of continuous epimorphisms $\varphi_N : Q \longrightarrow Q/N$ with $N = \mathrm{Ker}(\varphi_N)$ is finite. Therefore for any finite group A, the number n_A of open normal subgroups N of Q with $Q/N \cong A$ equals the number of continuous epimorphisms $Q \longrightarrow A$, which in turn equals m, because Q is an m-quasifree group (just put $B = 1$ in the embedding problem). Now

$$w_0(Q) = \sum_A n_A = m\aleph_0 = m,$$

since the number of isomorphism classes of finite groups is $\aleph_0$. □

Let $\mathcal{C}$ be a variety of finite groups. If G is a profinite group, define $R_{\mathcal{C}}(G)$ to be the intersection of all closed normal subgroups N of G such that $G/N \in \mathcal{C}$. Then $G/R_{\mathcal{C}}(G)$ is the maximal pro-$\mathcal{C}$ quotient of G (see [RZ], Section 3.4). The following result is clear.

Proposition 1.3 *Let $\mathcal{C}' \subseteq \mathcal{C}$ be varieties of finite groups, and let m be an infinite cardinal. If Q is an m-quasifree pro-$\mathcal{C}$ group, then its maximal pro-$\mathcal{C}'$ quotient $Q/R_{\mathcal{C}'}(Q)$ is an m-quasifree pro-$\mathcal{C}'$ group.*

Proposition 1.4 *Let G be an m-quasifree pro-$\mathcal{C}$ group. Then G contains a free pro-$\mathcal{C}$ group of countable rank.*

Proof. We observe (see [RZ], Corollary 2.6.6) that if H is a pro-$\mathcal{C}$ that admits a a countable set of generators converging to 1, then H contains a countable collection of open normal subgroups

$$H = U_0 > U_1 > \cdots$$

that form a fundamental system of neighborhoods of 1, and so

$$H = \varprojlim_{i \in I} H/U_i \leq \prod_i H/U_i.$$

It follows that H appears as a closed subgroup of the catesian product of the set of all finite groups in $\mathcal{C}$. In particular the free pro-$\mathcal{C}$ group F of countable rank appears as a closed subgroup of such cartesian product.

Therefore to prove the proposition it is enough to construct an epimorphism $\lambda : G \longrightarrow \prod_{i=0}^{\infty} K_i$, where K_i runs over all finite groups in $\mathcal{C}$, where we assume $K_0 = 1$. To do this we construct inductively compatible epimorphisms

$$\lambda_n : G \longrightarrow \prod_{i=0}^{n} K_i.$$

If λ_{n-1} has been constructed consider the following split embedding problem

$$\begin{array}{ccc} & & G \\ & & \downarrow{\scriptstyle \lambda_{n-1}} \\ \prod_{i=0}^{n} K_i & \xrightarrow{\alpha_n} & \prod_{i=1}^{n-1} K_i \end{array},$$

where α is the natural projection. Since G is quasifree, there exists an epimorphism $\lambda_n : G \longrightarrow \prod_{i=0}^{n} K_i$ such that $\alpha_n\lambda_n = \lambda_{n-1}$ $(n = 1, 2, \ldots)$. The inverse limit of these maps

$$\lambda = \varprojlim{}_n \lambda_n : G \longrightarrow \prod_{i=0}^{\infty} K_i$$

provides the required epimorphism. □

If A and B are pro-$\mathcal{C}$ groups, we denote by $A \amalg B$ their free pro-$\mathcal{C}$ product, i.e., their coproduct in the category of pro-$\mathcal{C}$ groups (see [RZ], Section 9.1). For simplicity we state the following lemma only for finite groups A and B, but the result is valid in general. One says that a variety of finite groups $\mathcal{C}$ is *extension closed* if whenever $1 \to K \to G \to H \to 1$ is an exact sequence of finite groups such that $K, H \in \mathcal{C}$, then $G \in \mathcal{C}$.

Lemma 1.5 *Assume that the variety of finite groups $\mathcal{C}$ is extension closed. Let $G = A \amalg B$ be a free pro-$\mathcal{C}$ product of two pro-$\mathcal{C}$ groups A and B. Let A^G denote the smallest closed normal subgroup of G generated by A. Then A^G is the free pro-$\mathcal{C}$ product of the subgroups $\{A^b = b^{-1}Ab \mid b \in B\}$ of G.*

Proof: This follows from the analogous of the Kurosh subgroup theorem for free products of pro-$\mathcal{C}$ groups. Indeed, observe that $K = A^G$ is a normal open subgroup of $G = A \amalg B$ with $G/K \cong B$. Then (see [RZ], Theorem 9.1.9)

$$K = \left[\coprod_{\tau \in K\backslash G/A} K \cap g_\tau A g_\tau^{-1} \right] \amalg \left[\coprod_{\tau \in K\backslash G/B} K \cap g'_\nu B g'^{-1}_\nu \right],$$

where g_τ ranges through a set of representatives of the double cosets $K\backslash G/A$ and g_ν ranges through a set of representatives of the double cosets $K\backslash G/B$, and where F is a free pro-$\mathcal{C}$ group of rank $1 + [G : K] - |K\backslash G/A| - K\backslash G/B$. In our case, since $K \triangleleft G$, $K \geq A$ and $G/K \cong B$, it follows that rank $(F) = 0$ and $K = \coprod_{b \in B} A^b$, the free pro-$\mathcal{C}$ product of the conjugates A^b of A by the elements of B. □

Examples 1.6

1. A free profinite group $F = F(m)$ of rank m is m-quasifree. In fact a profinite group is free profinite of rank m if and only if it is m-quasifree and projective.

2. (D. Haran) If Q is an m-quasifree group and H is a profinite group with $d(H) \leq m$, then their free profinite product $Q \amalg H$ is m-quasifree.

3. Let F be a free profinite group on a countable set of generators $x_1, y_1, x_2, y_2, \ldots$ convergent to 1. Observe that the infinite product $[x_1, y_1][x_2, y_2] \cdots$ converges in F and so it defines a unique element r. Define a profinite group G imposing on F the relation $[x_1, y_1][x_2, y_2] \cdots$, i.e., $G = F/(r)$, where (r) denotes the smallest closed normal subgroup of F containing r.

We shall show that G is $\aleph_0$-quasifree. Consider a split embedding problem

$$\begin{array}{ccc} & & G \\ & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{\alpha} & B \end{array}$$

Put K =Ker(α). Let $\theta : B \longrightarrow A$ be a homomorphism such that $\alpha\theta = \mathrm{id}_B$. Then $A = K \rtimes \theta(B)$. Since B is finite, there exists a natural number t such that $f(x_j) = f(y_j) = 1$, for all $j > t$. Let $k_1, \ldots, k_n$ be the elements of K.

Next we define an infinite countable set of continuous epimorphism $\{\eta_s : F \longrightarrow A \mid s = 0, 1.2, \ldots\}$. The epimorphism η_s is determined by

$$\eta_s(x_j) = \begin{cases} (\theta f)(x_j), & \text{if } 1 \leq j \leq t + s; \\ k_i, & \text{if } j = t + s + i, i = 1, \ldots, n; \\ 1 & \text{if } j > t + s + n. \end{cases}$$

and

$$\eta_s(y_j) = \begin{cases} (\theta f)(y_j), & \text{if } 1 \le j \le t+s; \\ k_i, & \text{if } j = t+s+i, i = 1, \dots, n; \\ 1 & \text{if } j > t+s+n. \end{cases}$$

Observe that $\eta_s(r) = 1$. Therefore, η_s induces a continuous epimorphism $\lambda_s : G \longrightarrow A$. Moreover $\lambda_s \neq \lambda_{s'}$, if $s \neq s'$, and $\alpha\lambda_s = f$, for all $s = 0, 1, 2, \dots$. Since $d(G) = \aleph_0$, this shows that the above embedding problem has exactly $\aleph_0$ solutions.

4. It is easy to generalize Example 3 to an infinite family of examples of the same type, for example by setting $r = \prod_{i=1}^{\infty} [x_i^2, y_i^2]$.

2. Open Subgroups of Quasifree Groups

Theorem 2.1 *Assume that $\mathcal{C}$ is an extension closed variety of finite groups. Let H be an open subgroup of an m-quasifree pro-$\mathcal{C}$ group G. Then H is m-quasifree.*

Proof. Consider the following finite split embedding problem of pro-$\mathcal{C}$ groups for H

$$\begin{array}{ccc} & & H \\ & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{\alpha} & B \end{array}$$

We shall prove first that this embedding problem has at least one solution.

Put $K = \mathrm{Ker}(\alpha)$, and let $T = \mathrm{Ker}(f)_G$ denote core of the subgroup $\mathrm{Ker}(f)$ in G, that is, the intersection of all conjugates of $\mathrm{Ker}(f)$ in G. Then T is open and normal in G. Let $\beta : G \longrightarrow B' = G/T$ be the canonical epimorphism and define $B'_H = \beta(H) = H/T$. Denote by

$$\bar{f} : B'_H = H/T \longrightarrow B$$

the natural map induced by f.

Construct the free pro-$\mathcal{C}$ product $A' = K \amalg B'$ of K and B'. By Lemma 1.5, the closed normal subgroup $\tilde{K}$ of A' generated by K is the free profinite product $\tilde{K} = \coprod_{b \in B'} K^b$. Note that

$$A' = K \amalg B' = \tilde{K} \rtimes B'.$$

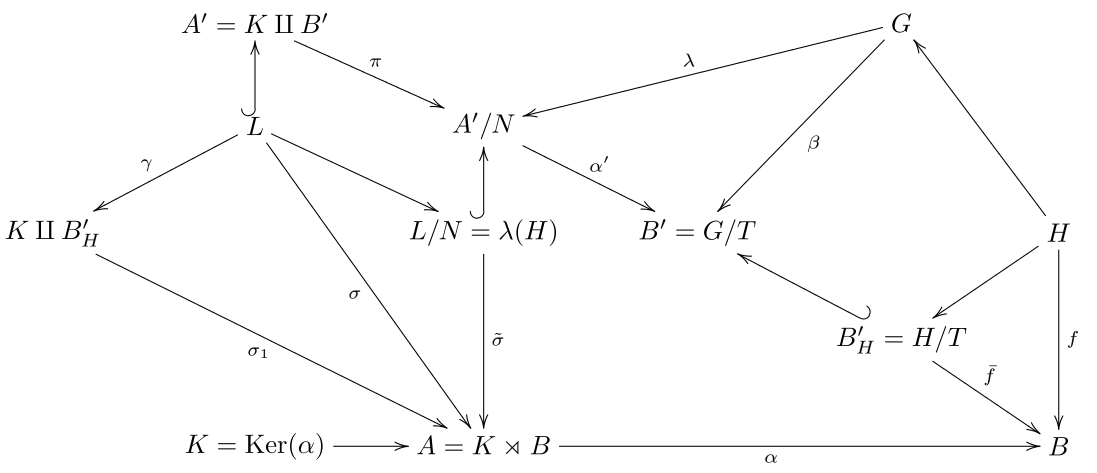
Consider the open subgroup

$$L = \tilde{K} \rtimes B'_H$$

of A'.

Observe that the subgroup B'_H of L normalizes the free factors $\coprod_{b \in B'_H} K^b$ and $\coprod_{b \in (B' - B'_H)} K^b$ of $\tilde{K} = \coprod_{b \in B'} K^b = (\coprod_{b \in B'_H} K^b) \amalg (\coprod_{b \in (B' - B'_H)} K^b)$. It follows from Lemma 1.5 that the closed normal subgroup of $\tilde{K}$ generated by $\coprod_{b \in (B' - B'_H)} K^b$ is normalized by B'_H, and therefore it is normal in L. Thus there is a natural epimorphism

$$\gamma : L \longrightarrow (\coprod_{b \in B'_H} K^b) \rtimes B'_H = K \amalg B'_H.$$



Let

$$\sigma_1 : K \amalg B'_H \longrightarrow A = K \rtimes B$$

be the continuous epimorphism induced by the identity map $K \longrightarrow K$ and the map $\bar{f} : B'_H \longrightarrow B$. Put $\sigma = \sigma_1\gamma$.

Define

$$N = (\mathrm{Ker}(\sigma) \cap \tilde{K})_{A'},$$

the core of $\mathrm{Ker}(\sigma) \cap \tilde{K}$ in A'; so that N is open normal in A' and contained in $\mathrm{Ker}(\sigma) \cap \tilde{K}$. Observe that $N \cap B' = N \cap K = 1$. Consider the finite group

$$A'/N = (\tilde{K}/N) \rtimes B'.$$

Let

$$\pi : A' \longrightarrow A'/N \quad \text{and} \quad \alpha' : A'/N \longrightarrow B'$$

be the the canonical epimorphisms. Since G is m-quasifree and α' is an epimorphism of finite groups which splits, there exists an epimorphism

$$\lambda : G \longrightarrow A'/N$$

such that $\alpha'\lambda = \beta$. Since $N \leq \mathrm{Ker}(\sigma)$, we deduce that σ factors through $L/N = \pi(L)$. Let $\tilde{\sigma} : L/N \longrightarrow A$ be the map induced by σ.

We claim that $L/N = \lambda(H)$. To see this it suffices to show that $\pi^{-1}(\lambda(H)) = L$. We show first that

$$\lambda(H) = \alpha'^{-1}(\beta(H)).$$

Since $\beta(H) = \alpha'(\lambda(H))$, we clearly have that $\lambda(H) \leq \alpha'^{-1}(\beta(H))$. For the reverse inclusion, note that

$$[G : H] \geq [\lambda(G) : \lambda(H)] \geq [\alpha'^{-1}(B') : \alpha'^{-1}(\beta(H))] = [B' : \beta(H)] = [B' : B'_H] = [G : H].$$

Hence

$$\lambda(H) = \alpha'^{-1}(\beta(H)) = (\tilde{K}/N) \rtimes B'_H,$$

as desired. Therefore,

$$\lambda(H) \geq \mathrm{Ker}(\alpha') = \tilde{K}/N,$$

and so,

$$\pi^{-1}(\lambda(H)) \geq \tilde{K}.$$

Since obviously $\pi^{-1}(\lambda(H)) \geq B'_H$, we deduce that $\pi^{-1}(\lambda(H)) \geq L = \tilde{K} \rtimes B'_H$. If $\pi^{-1}(\lambda(H)) \neq L$, then $\pi^{-1}(\lambda(H))$ would contain elements of $B' - B'_H$, and so

$$B'_H = \beta(H) = (\alpha'\lambda)(H) = (\alpha'\pi\pi^{-1}\lambda)(H) \neq B'_H,$$

a contradiction. Thus $\pi^{-1}(\lambda(H)) = L$, proving the claim.

Next define $\lambda' = \tilde{\sigma}\lambda_{|H}$. We check now that $\alpha\lambda' = f$. Indeed, $\alpha'(L/N) \leq B_{H'}$. On the other hand, since $L = \tilde{K} \rtimes B'_H$ we have $L/N = (\tilde{K}/N) \rtimes B'_H$ Applying $\tilde{\sigma}$ to B'_H and K/N as subgroups of L/N, we get $\tilde{\sigma}_{|B'_H} = \bar{f}$ and $\tilde{\sigma}(K/N) \leq K$. Thus, $\alpha\tilde{\sigma} = \bar{f}\alpha'_{|L/N}$. Hence $\alpha\lambda' = \bar{f}\alpha'\lambda_{|H} = \bar{f}\beta_{|H} = f$, as needed.

To finish the proof that H is m-quasifree, we must verify that the above split embedding problem has exactly m solutions. The number of maps λ in the diagram above is m, since G is m-quasifree. Since m is infinite and the index of H in G is finite, the number of λ' that can be obtained by the construction above is m. So the total number of solutions of the diagram

$$\begin{array}{ccc} & & H \\ & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{\alpha} & B \end{array}$$

is at least m. But obviously the total number of solutions is at most $d(H) = d(G)$. By Lemma 1.2, $d(G) = m$. Thus the total number of solutions is m. □

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